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STUDIES IN MATHEMATICS

CONVERGENCE OF FOURIER SERIES

HUBERT E. BRAY

UNIFORM CONVERGENCE OF FOURIER SERIES

JOHN P. NASH

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CONVERGENCE OF FOURIER SERIES

1. Introduction. This article deals with some new aspects of the question of the convergence and of the rapidity of convergence of the Fourier series $\mathfrak{S}(f)$ of a function $f(x) \in (L)$. It deals entirely with simple convergence, not with summability. In fact the methods employed have little connection with summability for the reason that they are designed to exploit whatever oscillatory character the function $f(x)$ may have, and especially that of the Dirichlet kernel, which is not of fixed sign, whereas the kernels of summability methods are usually of positive type. On this account frequent use is made of the Second Mean Value Theorem in estimating the magnitudes of the integral expressions which occur in the discussion, and thus our results are obtained without resorting to the cruder device of inserting absolute value signs under the sign of integration, even in intervals in which the integrals in question are absolutely convergent.

We are concerned primarily with the behavior of integrals of the form

$$\int_\lambda^\mu (\phi(t)/t) \sin ntdt, \quad \int_\lambda^\mu (\phi(t)/t) \cos ntdt, \qquad (0 < \lambda < \mu \leqq \pi)$$

which are essentially the types met with in the study of the Fourier series $\mathfrak{S}(f)$ and the conjugate series $\overline{\mathfrak{S}}(f)$, respectively, corresponding to the function $f(x)$ belonging to the class (L). Our object is to study integrals of these types in order to formulate general criteria for the convergence of one or the other or both of these series.

The classical criteria depend to a large extent on the absolute convergence properties of Lebesgue integrals. For example, that of Lebesgue, regarded as one of the more general, involves the (absolute) convergence property

$$\text{(L)} \qquad \int_\delta^\pi \left| (\phi(t) - \phi(t+\delta))/t \right| dt = o(1), \qquad (\delta \to 0)$$

which implies the convergence property

$$\text{(F)} \qquad \int_{\pi/n}^{\pi} (\phi(t)/t) \sin ntdt = o(1) \qquad (n \to \infty)$$

provided that $|\phi(t)|$ satisfies a simple additional condition as $t \to 0$. The property (F) in turn implies the convergence of $\mathfrak{S}(f)$ to $f(x_0)$ at the point x_0, if $\phi(t)$, by definition, is given by

$$\phi(t) = f(x_0 + t) + f(x_0 - t) - 2f(x_0).$$

But the condition (L) imposes a restriction on the behavior of $\phi(t)$ which can be shown to be, in a quite definite sense, very severe. For example it is not difficult to prove that if $f(x) = \cos(1/x)$, $x \neq 0$, $x \in [-\pi, \pi]$, then

$$s_n(0) = (1/\pi) \int_0^{\pi} (\phi(t)/2 \sin (t/2)) \sin (n + 1/2)tdt = O(n^{-1/4}),$$

which implies that $\mathfrak{S}(f)$ converges to zero fairly rapidly at $x = 0$; whereas the function $\phi(t)$, in the example, does not satisfy condition (L) at all. This is due of course to the presence in (L) of the bars under the integral sign. Examples of this kind are easy to find and are exhibited in §5.

Throughout the paper we shall assume that the function $f(x)$ is of period 2π, is integrable (Lebesgue) in every interval, and satisfies at the point x_0 the condition

$$\text{(A}_1\text{)} \qquad \lim_{\mu=0} (1/\mu) \int_0^{\mu} \phi(t)dt = 0$$

in other words, $f(x)$ is the symmetric derivative at x_0 of its own integral. This condition is satisfied almost everywhere and in any case is essential for our purposes. We will now describe one of the results established in the paper.

The condition (A_1), as is well known, implies that the integral

$$\int_0^{\pi/n} (\phi(t)/t) \sin ntdt = o(1) \qquad (n \to \infty).$$

A necessary and sufficient condition, under (A_1), in order that

$$s_n(x_0) - f(x_0)$$

$$= (1/\pi) \int_0^\pi (\phi(t)/2 \sin (t/2)) \sin (n + 1/2)t dt = o(1)$$

is therefore that

$$\int_{\pi/n}^\pi (\phi(t)/2 \sin (t/2)) \sin (n + 1/2)t dt = o(1)$$

and this, as is well known, is equivalent to:

$$I_n' = \int_{\pi/n}^\pi (\phi(t)/t) \sin nt dt = o(1), \qquad (n \to \infty)$$

where n varies continuously or by integers $1, 2, 3, \cdots$. It seems natural therefore to study the properties of integrals of the type I_n' and also of the type:

$$I_n'' = \int_{\pi/n}^\pi (\phi(t)/t) \cos nt dt,$$

where $\phi(t)$ satisfies condition (A_1). Results concerning these two types are obtained in §6 in studying conditions under which both $\mathfrak{S}(f)$ and $\overline{\mathfrak{S}}(f)$ converge at x_0. We shall not discuss them at this point but consider, by way of illustration, integrals of the types

$$I_n'(\lambda, \mu) = \int_\lambda^\mu (\phi(t)/t) \sin nt dt,$$

$$I_n''(\lambda, \mu) = \int_\lambda^\mu (\phi(t) dt) \cos nt dt,$$

$$I_n(\lambda, \mu) = iI_n'(\lambda, \mu) + I_n''(\lambda, \mu) \int_\lambda^\mu (\phi(t)/t) e^{int} dt,$$

where $\pi/n \leqq \lambda < \mu \leqq \pi$.

Obviously, a sufficient condition that $\mathfrak{S}(f)$ converge to $f(x_0)$ is that:

(a) $I_n(\lambda, \mu) = 0(1)$ *uniformly for all* $\lambda, \mu, \pi/n \leqq \lambda < \mu \leqq \pi$.

It is not difficult to prove, see §5, that condition (a) is considerably more general than that of Lebesgue, and therefore we are

interested in discovering its significance in terms of the intrinsic properties of the function $\phi(t)$. The theorem we prove is the following one, in which we employ the notation:

$$S(F, \lambda, h, 2k) = (h/3)[F(\lambda) + 4F(\lambda + h) + 2F(\lambda + 2h) + \cdots + F(\lambda + 2kh)]$$

to represent the Simpson approximation to the integral:

$$\int_0^{2kh} F(\lambda + t)dt$$

THEOREM. *Let* $Q(t) = (1/t)\int_0^t \phi(t)dt$, $Q_1(t) = \int_0^t Q(t)dt$. *If* $Q(t) = o(1)$, *a necessary and sufficient condition that* $\phi(t)$ *satisfy condition* (a), *that is*:

(A) $I_n(\lambda, \mu) = o(1)$ *uniformly* $(\pi/n \leqq \lambda < \mu \leqq \pi)$

is that the function $Q(t)$ *satisfy the condition*:

(B) $$S(Q_1, \lambda, h, 2k) - \int_0^{2kh} Q_1(\lambda + t)dt = o(h^2)$$

uniformly as $h \to 0$, *for all intervals* $[\lambda, \lambda + 2kh]$ *such that* $0 \leqq \lambda < \lambda + 2kh \leqq 2\pi$.

By means of this theorem we can therefore establish the following general criterion for the convergence of $\mathfrak{S}(f)$:

If $\phi(t)$ *satisfies the conditions* (A_1) *and* (B) *then* $\mathfrak{S}(f)$ *converges at the point* x_0 *to* $f(x_0)$.

Our main results are expressed in terms of the order of approximation to certain integrals by means of elementary mechanical quadratures, namely the Trapezoid Rule and Simpson's Rule. We use the following notation, in which $F(x)$ represents a continuous function:

$$T(F, \lambda, h, k) = (h/2)[F(\lambda) + 2F(\lambda + h) + \cdots + 2F(\lambda + (k-1)h) + F(\lambda + kh)].$$

$$S(F, \lambda, h, 2k) = (h/3)[F(\lambda) + 4F(\lambda + h) + 2F(\lambda + 2h) + \cdots + 4F(\lambda + (2k-1)h) + F(\lambda + 2kh)].$$

$$E(F, \lambda, h, 2k) = F(\lambda) - 2F(\lambda + h) + 2F(\lambda + 2k) - \cdots - 2F(\lambda + (2k-1)h) + F(\lambda + 2kh).$$

The following relationships, of importance in the proofs, are listed for easy reference:

(I) $T(F, \lambda, 2h, k) - T(F, \lambda, h, 2k) = (h/2)E(F, \lambda, h, 2k)$

(II) $4T(F, \lambda, h, 2k) - T(F, \lambda, 2h, k) = 3S(F, \lambda, h, 2k)$

(III) $S(F, \lambda + h, h, 2k - 2) - S(F, \lambda, h, 2k)$
$= (h/3)[E(F, \lambda, h, 2k) - \{2F(\lambda) + F(\lambda + h)$
$+ F(\lambda + (2k - 1)h) + 2F(\lambda + 2kh)\}].$

These relationships, though perhaps not familiar, are easily verified.

2. Equivalence Theorems (A, C), (B, C). In this section we establish the main general theorems concerning the equivalence of the conditions (A), (B), (C), and also the conditions (A′), (B′), (C′), which we now define.

DEFINITION 1. *If $F(x)$ is a continuous function $0 \leqq x \leqq \pi$, and $F_1(x) = \int_0^t F(t)dt$, then*

(A) *$F(x)$ has the property* (A) $\sim$

$$\int_\lambda^\mu e^{inx}dF(x) = o(1) \qquad (n \to \infty)$$

uniformly for all λ, μ; $0 \leqq \lambda \leqq \mu \leqq 1$;

(B) *$F(x)$ has the property* (B) $\sim$

$$S(F_1, \lambda, h, 2k) - \int_\lambda^{\lambda+2kh} F_1(t)dt = o(h^2) \qquad (h \to 0)$$

uniformly for all λ, k such that $0 \leqq \lambda \leqq \lambda + 2kh \leqq 1$;

(C) *$F(x)$ has the property* (C) $\sim$

$$E(F_1, \lambda, h, 2k) - (h/2)(F(\lambda + 2kh) - F(\lambda)) = o(h) \qquad (h \to 0)$$

uniformly for all λ, k, $0 \leqq \lambda \leqq \lambda + 2kh \leqq 1$.

The conditions (A′), (B′), (C′), are like (A), (B), (C), except that the interval of integration is of length 2π and $F(x)$ is assumed to be equal to a linear function plus a function of period 2π.

DEFINITION 2. *If* $F(x)-mx$ *is a periodic continuous function, of period* 2π, *then*

(A′) $F(x)$ *has the property* (A′) $\sim$

$$\int_0^{2\pi} e^{inx}dF(x) = o(1) \qquad (n = \text{integer}),\ (n \to \infty);$$

(B′) $F(x)$ *has the property* (B′) $\sim$

$$S(F_1, \lambda, h, 2n) - \int_0^{2\pi} F_1(\lambda + t)dt = o(h^2), \qquad (h = \pi/n \to 0),$$

uniformly for all λ, $0 \leqq \lambda \leqq 2\pi$;

(C′) $F(x)$ *has the property* (C′) $\sim$

$$E(F_1, \lambda, h, 2n) - (h/2)(F(\lambda+2\pi) - F(\lambda)) = o(h), \qquad (h = \pi/n \to 0)$$

uniformly for all λ, $0 \leqq \lambda \leqq 2\pi$.

THEOREM (A, C). *If* $F(x)$ *is a continuous function,* $0 \leqq x \leqq \pi$, *and* $F_1(x) = \int_0^x F(t)dt$, *a necessary and sufficient condition that*

(A) $$\int_\lambda^\mu e^{inx}dF(x) = o(1)$$

uniformly for all λ, μ, $(0 \leqq \lambda < \mu \leqq 1)$ *is that*

(C) $$E(F_1, \lambda, h, 2k) - (h/2)\left[F(\lambda+2kh) - F(\lambda)\right] = o(h), \qquad (h \to 0)$$

uniformly for all λ, k *where* k *is a positive integer such that* $0 \leqq \lambda < \lambda + 2kh \leqq 1$, *and* $h > 0$.

Suppose first that (A) is satisfied, then

$$\int_0^{2k\pi/n} \cos nx dF(\lambda + x) = o(1)$$

and on integrating by parts and dividing by n, setting $h = \pi/n$, we have

$$(2.1) \qquad \int_0^{2kh} F(\lambda + x) \sin nx dx = \left[F(\lambda) - F(\lambda + 2kh)\right]/n + o(h).$$

We now consider the Fourier series of the odd function $p(x)$, of period 2π,

$$p(x) = \text{signum}(\sin x) = (4/\pi)\sum_{j=0}^{\infty} (\sin(2j+1)x)/(2j+1),$$

with $p(0) = p(\pi) = 0$. On replacing x by nx in this formula, multiplying by $F(\lambda+x)$, and integrating termwise, as we may, then applying (2.1) we obtain:

$$\begin{aligned}
&\int_0^{2kh} F(\lambda + x)p(nx)dx \\
&\quad = (4/\pi)\sum_{j=0}^{\infty}\int_0^{2kh} F(\lambda + x)(\sin(2j+1)nx/(2j+1))dx \\
&\quad = (4/\pi)(F(\lambda) - F(\lambda + 2kh))\sum_{j=0}^{\infty} (1/n(2j+1)^2) \\
&\qquad + (4/\pi)\sum_{j=0}^{\infty} o(1/n(2j+1)^2) \\
&\quad = (h/2)[F(\lambda) - F(\lambda + 2kh)] + o(h),
\end{aligned}$$

since $\sum_0^{\infty} (2j+1)^{-2} = \pi^2/8$. This result is equivalent to

$$-E(F_1, \lambda, h, 2k) = (h/2)[F(\lambda) - F(\lambda + 2kh)] + o(h),$$

which is equivalent to (C).

Conversely, to prove (A), assuming (C), we consider the integral, in which $h = \pi/n$, namely

$$\begin{aligned}
\int_0^{2kh} \sin nt\, dF(\lambda + t) &= -n\int_0^{2kh} F(\lambda + t)\cos ntdt \\
&= -n[F_1(\lambda + 2kh) - F_1(\lambda)] \\
&\quad - n^2\int_0^{2kh} F_1(\lambda + t)\sin ntdt.
\end{aligned}$$

The last expression is obtained by integrating twice by parts. On dividing it by n^2 we see that we have to prove that

$$(2.2)\quad \int_0^{2kh} F_1(\lambda + t)\sin ntdt + [F_1(\lambda + 2kh) - F_1(\lambda)]/n = o(h^2)$$

uniformly for all $\lambda>0$ such that $\lambda+2kh\leqq 1$. By the definition of the symbol E and by the periodicity of the function sin nt, the left members of (2.2) is equal to:

$$(1/2)\int_0^{\pi/n} E(F_1, \lambda+t, h, 2k)\sin ntdt$$
$$+(1/2)\int_0^{\pi/n} F_1(\lambda+t)\sin ntdt$$
$$-(1/2)\int_0^{\pi/n} F_1(\lambda+t+2kh)\sin ntdt$$
$$+[F_1(\lambda+2kh)-F_1(\lambda)]/n.$$

Now, using (C), this expression is equal to:

$$(-\pi/4n)\int_0^{\pi/n}[F(\lambda+t)-F(\lambda+t+2kh)]\sin ntdt$$
$$+(1/2)\int_0^{\pi/n}[F_1(\lambda+t)-F_1(\lambda)]\sin ntdt$$
$$-(1/2)\int_0^{\pi/n}[F_1(\lambda+t+2kh)-F_1(\lambda+2kh)]\sin ntdt+o(h^2).$$

But since $F(t)$ is continuous, we have

$$F(\lambda+t)=F(\lambda)+o(1),\quad F(\lambda+t+2kh)=F(\lambda+2kh)+o(1)$$
$$F_1(\lambda+t)-F_1(\lambda)=t[F(\lambda)+o(1)]$$
$$F_1(\lambda+t+2kh)-F_1(\lambda+2kh)=t[F(\lambda+2kh)+o(1)]$$

uniformly as $h\to 0$, $0\leqq t\leqq h=\pi/n$. By inserting these three results in the three integrals respectively we find that the left member of (2.2) has been reduced to

$$-(\pi/2n^2)[F(\lambda)-F(\lambda+2kh)]$$
$$+(1/2)[F(\lambda)-F(\lambda+2kh)]\int_0^{\pi/n} t\sin ntdt+o(h^2)=o(h^2).$$

We have thus proved that condition (C) implies condition (A) in the form

$$\int_0^{2kh\lambda} \sin ntdF(\lambda + t) = o(1)$$

uniformly for all positive values of λ, k such that $\lambda + 2kh \leqq 1$, and this is equivalent to (A), since (A) is equivalent to

$$\int_0^{\mu} \sin nxdF(\lambda + x) = o(1), \qquad \textit{uniformly } (\lambda < \lambda + \mu \leqq 1)$$

and μ can be expressed as $2kh \pm l$ where $0 \leqq l \leqq 2\pi/n$. For an interval of length $l \leqq 2\pi/n$ may be ignored on account of the continuity of F and the fact that $\sin nx$ and $\cos nx$ are uniformly bounded and of bounded variation in all intervals of length $\leqq 2\pi/n$. This follows by a simple application of the Second Mean Value Theorem. The proof is therefore complete.

Two corollaries of this theorem are worthy of notice:

COROLLARY I (A, C), *in which the quantity* $o(1)$ *in* (A) *is replaced by* $O(h^{-\alpha})$, $(0 \leqq \alpha < 1)$, *and the quantity* $o(h)$ *in* (C) *is replaced by* $O(h^{1-\alpha})$.

COROLLARY II (A, C), *in which* $o(1)$ *in* (A) *is replaced by* $O(\rho(h) + \omega_F(h))$ *and* $o(h)$ *in* (C) *is replaced by* $O[h(\rho(h) + \omega_F(h))]$ *where* $\rho(h)$ *is a function which decreases to zero with* h, *and* $\omega_F(h)$ *is the well-known oscillation function associated with* F.

The reader will have little difficulty in verifying these corollaries by noting that with these changes the proofs go through step by step.

We can also state:

THEOREM (A′, C′) *If* $F(x) - mx$ *is a continuous function of period* 2π, *a necessary and sufficient condition that:*

(A′) $$\int_0^{2\pi} e^{inx}dF(x) = o(1) \quad \textit{as the integer } n \to \infty$$

is that

(C′) $$E(F_1, \lambda, h, 2n) - (h/2)[F(\lambda + 2\pi) - F(\lambda)] = o(h) \qquad (h = \pi/n \to 0)$$

uniformly for all λ, *as the integer* n *becomes infinite.*

The proof of this theorem is simpler than that of Theorem (A, C). For if $F(x)$ itself is periodic the proof can be carried out, step by step, in the same way except for simplifications due to the periodicity. If $F(x)$ is linear, in which case $F_1(x)$ is quadratic, the proof is immediate—the second members of (A′) and (C′) being equal to zero. With these remarks the reader can easily verify the theorem.

THEOREM (B, C). *If* $F(x)$ *is continuous,* $0 \leqq x \leqq \pi$, *and* $F_1(x) = \int_0^x F(t)dt$, *a necessary and sufficient condition that*

$$\text{(B)} \qquad S(F_1, \lambda, h, 2k) - \int_\lambda^{\lambda+2kh} F_1(t)dt = o(h^2)$$

uniformly for all λ, k *such that* $0 \leqq \lambda < \lambda + 2kh \leqq 1$, *is that:*

$$\text{(C)} \qquad E(F_1, \lambda, h, 2k) - (h/2)[F(\lambda + 2kh) - F(\lambda)] = o(h)$$

uniformly for all λ, k *as in* (B).

Proof. Assuming that (B) is valid, and replacing λ by $\lambda + h$, $2k$ by $2k-2$, we have also

$$S(F_1, \lambda + h, h, 2k - 2) - \int_{\lambda+h}^{\lambda+(2k-1)h} F_1(t)dt = o(h^2)$$

and on subtracting the members of (B) from the members of this equation, and using (§1, III), we obtain

$$(h/3)\big[E(F_1, \lambda, h, 2k) - \{2F_1(\lambda) + F_1(\lambda + h) + F_1(\lambda + (2k-1)h) + 2F_1(\lambda + 2kh)\}\big] + \int_0^h F_1(\lambda + t)dt + \int_0^h F_1(\lambda + (2k-1)h + t)dt = o(h^2),$$

or, on combining the terms in brackets with the two integrals,

$$\begin{aligned} &(h/3)E(F_1, \lambda, h, 2k) \\ &\quad + (1/3)\int_0^h [3F_1(\lambda + t) - 2F_1(\lambda) - F_1(\lambda + h)]dt \\ &\quad + (1/3)\int_0^h [3F_1(\lambda + (2k-1)h + t) \end{aligned} \tag{2.3}$$

$$- F_1(\lambda + (2k-1)h) - 2F_1(\lambda + 2kh)]dt = o(h^2).$$

Here we make a substitution in the first integral of (2.3):

$$\begin{aligned} 3F_1(\lambda + t) - 2F_1(\lambda) - F_1(\lambda + h) \\ = 3[F_1(\lambda + t) - F_1(\lambda)] - [F_1(\lambda + h) - F_1(\lambda)] \\ = 3t[F(\lambda) + o(1)] - h[F(\lambda) + o(1)] \end{aligned}$$

which follows from the continuity of F. It is easily seen that the first integral in (2.3) is equal to:

$$(1/3)(3h^2/2 - h^2)(F(\lambda) + o(1) = (h^2/6)(F(\lambda) + o(1)).$$

By treating the second integral of (2.3) in a similar way, we obtain:

$$(h/3)\{E(F_1, \lambda, h, 2k) + (h/2)[F(\lambda) - F(\lambda + 2kh)]\} = o(h^2);$$

and this is equivalent to (C).

Conversely, assuming (C), we have, by (§1, I).

$$\begin{aligned} T(F_1, \lambda, 2h, k) - T(F_1, \lambda, h, 2k) \\ = (h/2)E(F_1, \lambda, h, 2k) = (h^2/4)[F(\lambda + 2kh) - F(\lambda)] + o(h^2). \end{aligned}$$

We now iterate this formula, replacing h in succession by $h/2$, $h/4, \cdots h/2^\nu$, and sum. Since $h^2 \sum_1^\infty 4^{-\nu} = h^2/3$, we obtain from the right hand members the quantity $(h^2/3)[F(\lambda+2kh) - F(\lambda)] + o(h^2)$. And since the trapezoidal sum $T(F_1, \lambda, h/2^\nu, 2^{\nu+1}k)$ converges to $\int_0^{2kh} F_1(\lambda+t)dt$ the result is:

$$\begin{aligned} (2.4) \quad T(F_1, \lambda, 2h, k) - \int_0^{2kh} F_1(\lambda + t)dt \\ = (h^2/3)[F(\lambda + 2kh) - F(\lambda)] + o(h^2) \end{aligned}$$

and therefore if we replace h by $h/2$, k by $2k$, in this equation, and multiply both members by four:

$$\begin{aligned} (2.5) \quad 4T(F_1, \lambda, h, 2k) - 4\int_0^{2kh} F_1(\lambda + t)dt \\ = (h^2/3)[F(\lambda + 2kh) - F(\lambda)] + o(h^2). \end{aligned}$$

On subtracting members of (2.4) from those of (2.5) we obtain, since, by (§1, II),

$$4T(F_1, \lambda, h, 2k) - T(F_1, \lambda, 2h, k) = 3S(F_1, \lambda, h, 2k),$$

the final result

$$3S(F_1, \lambda, h, 2k) - 3\int_0^{2kh} F_1(\lambda + t)dt = o(h^2)$$

which is equivalent to (B).

This theorem, like Theorem (A, C), has two corollaries:

COROLLARY I (B, C): *in which the quantity* $o(h^2)$ *in* (B) *is replaced by* $O(h^{2-\alpha})$, *where* $0 \leqq \alpha < 1$, *and the quantity* $o(h)$ *in* (C) *is replaced by* $O(h^{1-\alpha})$.

COROLLARY II (B, C): *in which the quantity* $o(h^2)$ *in* (B) *is replaced by* $O[h^2(\rho(h) + \omega_F(h))]$ *and the quantity* $o(h)$ *in* (C) *is replaced by* $O[h(\rho(h) + \omega_F(h))]$, *where* $\rho(h)$ *is a function which decreases to zero with* h, *and* $\omega_F(h)$ *is the oscillation function associated with* F.

The reader can easily verify the corollaries by following through the steps of the proof of Theorem (B, C) with the indicated changes.

THEOREM (B′, C′) *If* $F(x) - mx$ *is a continuous function, of period* 2π, *a necessary and sufficient condition in order that, as the integer* n *becomes infinite,*

$$(B') \qquad S(F_1, \lambda, h, 2n) - \int_0^{2\pi} F_1(\lambda + t)dt = o(h^2) \quad (h = \pi/n)$$

uniformly for all λ, $0 \leqq \lambda \leqq 2\pi$, *is that*

$$(C') \qquad E(F_1, \lambda, h, 2n) - (h/2)(F(\lambda + 2\pi) - F(\lambda)) = o(h)$$

uniformly for all λ, $0 \leqq \lambda \leqq 2\pi$.

Here again it is unnecessary to give the details of the proof, which is like that of Theorem (B, C) but a good deal simpler on account of the periodicity. The proof is immediate if $F(x)$

is linear, and $F_1(x)$ quadratic, for then it follows easily that the left members of (B′) and (C′) are equal to zero. With these remarks it is unnecessary to give a separate proof.

3. The Equivalence Theorem (a, A). In this section we prove that, under the assumption that $\phi(t)$ satisfies the condition (A_1), the conditions (a), (A), are equivalent; in other words that, under this assumption, if $Q(t)$ satisfies condition (A), where $Q(t) = (1/t)\int_0^t \phi(t)dt$, then $\phi(t)$ satisfies condition (a); and conversely. It will then follow from the result of §2 *that any one of the three conditions*:

$$Q(t) \text{ satisfies (A), (B), (C)}$$

is equivalent to the condition:

$$\phi(t) \text{ satisfies (a).}$$

We begin by proving the following

LEMMA. *A necessary and sufficient condition in order that*

$$\text{(a)} \qquad I_n(\pi/n, \lambda) = \int_{\pi/n}^{\lambda} (\phi(t)/t)e^{int}dt \qquad (\pi/n < \lambda \leqq \pi)$$

converge to zero uniformly with respect to λ, *as* $n \to \infty$, *is that*

$$\text{(b)} \qquad I_n(\pi/n, \lambda_n) = \int_{\pi/n}^{\lambda_n} (\phi(t)/t)e^{int}dt \qquad (\pi/n < \lambda_n)$$

converge to zero for every sequence $\{\lambda_n\}$, *converging to zero, such that* $\lambda_n > \pi/n$.

The necessity of the condition (b) is obvious.

Assuming that $\lim_{n\to\infty} I_n(\pi/n, \lambda_n) = 0$ for every such $\{\lambda_n\}$ we first prove that

$$\lim_{n\to\infty} I_n(\pi/n, \pi) = 0.$$

Suppose, on the contrary, that a sequence of integers $\{n_k\}$ exists, such that

$$|I_{n_k}(\pi/n_k, \pi)| > 2h > 0 \qquad (h = \text{const.})$$

Then that exists a positive sequence $\{\mu_k\}$, $\mu_k > \pi/n_k$, such that

$$(3.1) \qquad |I_{n_k}(\pi/n_k, \mu_k)| > h, \qquad |I_{n_k}(\mu_k, \pi)| > h.$$

We can, of course, suppose that $\{\mu_k\}$ converges. If $\lim_{k\to\infty} \mu_k = \mu > 0$ then for all intervals (a, b), $\mu \leqq a < b \leqq \pi$,

$$\lim_{n\to\infty} \int_a^b (\phi(t)/t) e^{int} dt = 0 \qquad \textit{uniformly},$$

for $\phi(t)/t$ is summable in the closed interval $[\mu/2, \pi]$, and therefore the Riemann-Lebesgue Theorem applies in this interval. But this contradicts (3.1). It follows therefore that $\lim_{k\to\infty} \mu_k = 0$, and we have

$$|I_{n_k}(\pi/n_k, \mu_k)| > h;$$

but this contradicts the hypothesis (b). We have therefore proved that

$$\lim_{n\to\infty} I_n(\pi/n, \pi) = 0.$$

In order to prove (a) suppose, on the contrary, that $I_n(\pi/n, \lambda)$ does not converge uniformly to zero. Then there exists a subsequence $\{n_k\}$, also a sequence $\{\mu_k\}$, together with a positive constant h such that

$$|I_{n_k}(\pi/n_k, \mu_k)| > h \qquad (\mu_k > \pi/n_k)$$

and since we can extract a convergent subsequence from the sequence $\{\mu_k\}$ we may as well suppose that $\{\mu_k\}$ itself converges to $\mu \geqq 0$.

If $\mu = 0$ we have a contradiction of (b). If $\mu > 0$ we have

$$\limsup_{k\to\infty} |I_{n_k}(\pi/n_k, \mu_k)| \geqq h$$

and since, as we have just proved,

$$\lim_{k\to\infty} |I_{n_k}(\pi/n_k, \pi)| = 0,$$

it follows that

$$\limsup_{k\to\infty} \left| I_{n_k}(\mu_k, \pi) \right| \geqq h.$$

But this is impossible since $\phi(t)/t$ is summable in the interval $[\mu/2, \pi]$ and the Riemann-Lebesgue Theorem again applies. The lemma is therefore proved.

With the aid of this lemma we can now prove

THEOREM (a, A). *If $\phi(t)$ satisfies condition* (A_1), *that is*

$$(A_1) \qquad Q(t) = (1/t)\int_0^t \phi(t)dt = o(1), \qquad (t \to 0)$$

a necessary and sufficient condition that

$$(a) \qquad \int_{\pi/n}^{\lambda} (\phi(t)/t)e^{int}dt = o(1) \qquad (n \to \infty)$$

uniformly for all λ, $\pi/n \leqq \lambda \leqq \pi$, *is that* $Q(t)$ *satisfy condition* (A) *in the form*:

$$(A) \qquad \int_{+0}^{\mu} Q'(t)e^{int}dt = \int_0^{\mu} e^{int}dQ(t) = o(1) \qquad (n \to \infty)$$

uniformly for all μ, $0 \leqq \mu \leqq \pi$.

Since $Q(t) = \int_{+0}^{t} Q'(s)ds$, and since sin nt, cos nt, are uniformly bounded and of uniformly bounded variation in the intervals $[0, \pi/n]$, and since $\phi(t)$ satisfies condition (A_1), we can apply the Second Mean Value Theorem to show that the integral

$$\int_0^{\alpha} Q'(t)e^{int}dt = o(1) \qquad \textit{uniformly for all } \alpha,\ 0 \leqq \alpha \leqq \pi/n,$$

and thus it follows that condition (A) is equivalent to the condition:

$$(\alpha) \qquad \int_{\pi/n}^{\mu} Q'(t)e^{int}dt = o(1)$$

uniformly for μ, $\pi/n \leqq \mu \leqq \pi$.

Hence, assuming (A) we obtain, after integration by parts in (α),

$$Q(t)e^{int}\Big]_{\pi/n}^{\mu_n} - in\int_{\pi/n}^{\mu_n} Q(t)e^{int}dt = o(1)$$

for all sequences $\{\mu_n\}$ converging to zero, with $\mu_n > \pi/n$.

But the first term in the preceding equation is $o(1)$, and therefore

$$n\int_{\pi/n}^{\mu_n} Q(t)e^{int}dt = o(1),$$

and therefore, by the Second Mean Value Theorem,

$$\int_{\pi/n}^{\mu_n} (Q(t)/t) \begin{matrix}\cos\\ \sin\end{matrix} ntdt = (n/\pi)\int_{\pi/n}^{\nu_n} Q(t) \begin{matrix}\cos\\ \sin\end{matrix} ntdt = o(1)$$

since $\{\nu_n\}$ converges to zero, $\pi/n \leqq \nu_n \leqq \mu_n$. But also we have $\phi(t)/t = Q(t)/t + Q'(t)$, and therefore we have shown that (α) and the last equation imply that

$$\int_{\pi/n}^{\mu_n} (\phi(t)/t)e^{int}dt = o(1)$$

for every sequence $\{\mu_n\}$, $\mu_n > \pi/n$, converging to zero. But by the preceding Lemma, this implies that

(a) $$\int_{\pi/n}^{\lambda} (\phi(t)/t)e^{int}dt = o(1)$$

uniformly, $\lambda > \pi/n$. Hence (A) implies (a).

Conversely, assuming (a), consider the integral:

$$\int_{\pi/n}^{\lambda} (Q(t)/t)e^{int}dt$$

$$= \int_{\pi/n}^{\lambda} e^{int}\left((1/t^2)\int_0^t \phi(s)ds\right)dt$$

$$= \int_{\pi/n}^{\lambda} e^{int}\left((1/t)\int_0^1 \phi(t\theta)d\theta\right)dt \qquad (s = t\theta)$$

$$= \int_0^1 d\theta \int_{\pi/n}^{\lambda} \phi(\theta t)e^{int}(dt/t)$$

$$= \int_0^1 d\theta \int_{\theta\pi/n}^{\theta\lambda} \phi(r)e^{inr/\theta}(dr/r) \qquad (r = \theta t)$$

$$= \int_0^1 d\theta \int_{\pi/m}^{\mu} \phi(r)e^{imr}(dr/r) \qquad m = n/\theta,\ \mu = \theta\lambda \leqq \lambda$$

and since, by (a), the inner integral is $o(1)$, uniformly for all $\mu > \pi/m$, as $m \to \infty$, $0 < \theta \leqq 1$, it follows that the condition (a) implies the condition:

$$\text{(c)} \qquad \int_{\pi/n}^{\lambda} (Q(t)/t)e^{int}dt = o(1) \qquad (n \to \infty)$$

uniformly for all $\lambda \geqq \pi/n$.

Now, recalling that

$$Q'(t) = \phi(t)/t - Q(t)/t$$

it follows that from (a) and (c) we may infer

$$(\alpha) \qquad \int_{\pi/n}^{\lambda} Q'(t)e^{int}dt = o(1)$$

uniformly for λ, $\pi/n \leqq \lambda \leqq \pi$, and since we have shown that, when $\phi(t)$ satisfies (A_1), the conditions (α) and (A) are equivalent, it follows that,

$$\text{(A)} \qquad \int_0^{\lambda} Q'(t)e^{int}dt = o(1)$$

uniformly for λ, $0 \leqq \lambda \leqq \pi$. Thus the theorem is proved.

As an immediate corollary of Theorem (a, A) we can state, in consequence of the equivalence of conditions (A) and (B) the following criterion for the convergence of $\mathfrak{S}(f)$.

THEOREM 1. *If $\phi(t)$ satisfies condition (A_1), that is:*

$$(A_1) \qquad Q(t) = (1/t)\int_0^t \phi(t)dt = o(1)$$

and if $Q(t)$ has the property (B), *that is:*

$$(B) \qquad S(Q_1, \lambda, h, 2k) - \int_0^{2kh} Q_1(\lambda + t)dt = o(h^2),$$

where $Q_1(t) = \int_0^t Q(t)dt$, *uniformly for all* λ, k, $0 \leqq \lambda < \lambda + 2kh \leqq \pi$, *then* $\mathfrak{S}(f)$ *converges at* $x = x_0$ *to* $f(x_0)$.

This is the theorem stated at the end of §1.

4. Simultaneous Convergence of $\mathfrak{S}(f)$, $\overline{\mathfrak{S}}(f)$. We now apply the results in §§2, 3 to the convergence of both $\mathfrak{S}(f)$ and $\overline{\mathfrak{S}}(f)$ at $x = x_0$. The quantities to be considered are expressed in our notation as follows:

$$(4.1) \quad s_n(x_0) - f(x_0) = (1/\pi) \int_0^{\pi} (\phi(t)/t) \sin ntdt + o(1)$$

$$(4.2) \quad \bar{s}_n(x_0) = -(1/\pi) \int_0^{\pi} (\psi(t)/2 \tan (t/2))(1 - \cos nt)dt + o(1)$$

where

$$\phi(t) = f(x_0 + t) + f(x_0 - t) - 2f(x_0)$$
$$\psi(t) = f(x_0 + t) - f(x_0 - t)$$

and $s_n(x_0)$, $\bar{s}_n(x_0)$ are the respective partial sums of $\mathfrak{S}(f)$ and $\overline{\mathfrak{S}}(f)$. Let now

$$\phi_r(t) = f(x_0 + t) - f(x_0)$$
$$\phi_l(t) = f(x_0 - t) - f(x_0),$$

so that

$$\phi(t) = \phi_r(t) + \phi_l(t)$$
$$\psi(t) = \phi_r(t) - \phi_l(t),$$

and let

$$Q^r(t) = (1/t) \int_0^t \phi_r(t)dt, \qquad Q^l(t) = (1/t) \int_0^t \phi_l(t)dt.$$

We will suppose that $\phi_r(t)$ and $\phi_l(t)$ satisfy the conditions:

$$(A_1) \qquad Q^r(t) = o(1), \qquad Q^l(t) = o(1), \qquad (t \to 0)$$

$$(A)\qquad \int_0^{\lambda} e^{int}dQ^r(t) = o(1), \quad \int_0^{\lambda} e^{int}dQ^l(t) = o(1), \qquad (n \to \infty)$$

uniformly for all λ, $0 \leqq \lambda \leqq \pi$.

The conditions (A_1), (A), are together equivalent, by §3, to (A_1) and the conditions:

$$\int_{\pi/n}^{\lambda} (\phi_r(t)/t)e^{int}dt = o(1), \qquad \int_{\pi/n}^{\lambda} (\phi_l(t)/t)e^{int}dt = o(1), \ (n \to \infty)$$

uniformly for all λ, $(\pi/n \leqq \lambda \leqq \pi)$. But since condition (A_1) is valid, we have, since $1-\cos nt = 2\sin^2(nt/2)$,

$$(4.3)\qquad \begin{aligned} &\int_0^{\pi/n} (\phi_r(t)/t) \sin ntdt = o(1), \\ &\int_0^{\pi/n} (\phi_r(t)/t)(1 - \cos nt)dt = o(1) \end{aligned}$$

and a corresponding condition with ϕ_r replaced by ϕ_l; the second equation in (4.3) being obtained by using the Second Mean Value Theorem in the form:

$$\int_0^{\pi/n} (\phi_r(t)/t) \sin^2 (nt/2)dt = \int_{\xi}^{\pi/n} (\phi_r(t)/t) \sin (nt/2)dt = o(1).$$

Hence we have from (4.1) and (4.2)

$$\begin{aligned} s_n(x_0) - f(x_0) &= (1/\pi) \int_{\pi/n}^{\pi} (\phi(t)/t) \sin ntdt + o(1) \\ &= o(1) \end{aligned}$$

and also

$$\begin{aligned} \bar{s}_n(x_0) + (1/\pi) &\int_{\pi/n}^{\pi} (\psi(t)/2 \tan (t/2))dt \\ &= (1/\pi) \int_{\pi/n}^{\pi} (\psi(t)/2 \tan (t/2)) \cos ntdt + o(1) \\ &= (1/\pi) \int_{\pi/n}^{\pi} (\psi(t)/t) \cos ntdt + o(1) \\ &= o(1). \end{aligned}$$

We have thus proved that if $\phi_r(t)$ and $\phi_l(t)$ satisfy conditions (A_1) and (A) then $s_n(x_0)$ converges to $f(x_0)$ and also that $\bar{s}_n(x_0)$ and the quantity

$$-(1/\pi)\int_{\pi/n}^{\pi}(\psi(t)/2\tan(t/2))dt$$

are equiconvergent in the strict sense.

But since conditions (A) and (B) are equivalent we have therefore proved:

THEOREM 2. *If $\phi_r(t)$ and $\phi_l(t)$ both satisfy the condition* (A_1) *and if $Q^r(t)$ and $Q^l(t)$ satisfy the condition*:

$$\text{(B)}\qquad S(Q_1^p,\lambda,h,2k)-\int_0^{2kh}Q_1^p(\lambda+t)dt=o(h^2)\quad(p=r,l)$$

uniformly $(0\leqq\lambda<\lambda+2kh\leqq\pi)$, *where* $Q^p(t)=(1/t)\int_0^t\phi_p(t)dt$, $Q_1^p(t)=\int_0^t Q^p(t)dt$, *then, as* $n\to\infty$,

$$s_n(x_0)-f(x_0)=o(1)$$

$$\bar{s}_n(x_0)+\int_{\pi/n}^{\pi}(\psi(t)/2\tan(t/2))dt=o(1).$$

5. Examples. In order to construct a continuous function which satisfies conditions (A_1) and (a) *but not* (L) we proceed as follows: let

$$\phi(t)=\psi(t)\sin(1/t)$$

where $\psi(t)$ is a continuous increasing function of t with $\psi(0)=\phi(0)=0$. We have to prove that the function $\psi(t)$ can be chosen so that the quantity

$$I_\delta=\int_\delta^1\left|\psi(t)\sin(1/t)-\psi(t+\delta)\sin(1/(t+\delta))\right|dt/t$$

is not $o(1)$ as $\delta\to0$. We make the substitution $t=1/s$, $N=1/\delta$, $\theta(s)=\psi(1/s)$ and obtain:

$$I_\delta = \int_1^N \theta(s) \left| \sin s - g(s, N) \sin s' \right| (1/s)ds$$

(5.1)

$$\geqq \theta(N) \int_1^N \left| \sin s - g(s, N) \sin s' \right| (1/s)ds,$$

where $g(s, N) = \psi(t+\delta)/\psi(t) \geqq 1$, $s' = Ns/(N+s)$. Let

$$p(s) = s - s' = s^2/(N + s).$$

This quantity is the difference of "phase" between sin s and sin s'.

Instead of the interval of integration $(1, N)$ we shall take a subinterval (a_N, b_N) such that (i), $p(a_N)$, $p(b_N)$ are integral multiples of 2π and (ii) $a_N \sim N^{2/3}$, $b_N \sim N^{3/4}$.

Now, since

$$p(s) = s^2/N - s^3/N^2 + \cdots$$
$$p'(s) = 2(s/N) - 3(s/N)^2 + \cdots = 2s/N + O(N^{-1/2})$$

it follows that if s is increased by the amount $(N\pi/s)h$ $p(s)$ will be increased by the amount:

$$\begin{aligned} & p'(s + \theta hN\pi/s) \cdot hN\pi/s && (0 < \theta < 1) \\ & \quad = 2(s + \theta hN\pi/s) \cdot h\pi/s + O(N^{-1/2}) \cdot N\pi h/s \\ & \quad = 2\pi h + \theta\pi^2 h^2 N/s^2 + O(N^{1/2})/s && (0 < h \leqq 1) \\ & \quad = 2\pi h + O(N^{-1/3}) + O(N^{-1/6}) \\ & \quad = 2\pi h + O(N^{-1/6}). \end{aligned}$$

In other words, in an interval of length $N\pi/s$, $p(s)$ will vary approximately linearly. Thus, if for a given value $s = s_1$, of the the form $s_1 = n_1\pi$, sin s' is approximately in phase with sin s, then, after s is increased to the value $s_2 = s_1 + N\pi/s_1$, sin s' will again be approximately in phase with sin s. In approximately one third of the half-period intervals $(n\pi, (n+1)\pi)$ contained in (s_1, s_2) s' will lag behind s by the amount p such that

$$(2k + 1)\pi - \pi/3 \leqq p \leqq (2k + 1)\pi + \pi/3.$$

These intervals will be situated very nearly in the middle third of the interval (s_1, s_2). Moreover in the middle third of each one of these half-period intervals, where $|\sin s| > \sqrt{3}/2$, $\sin s$ and $\sin s'$ will have opposite signs and therefore the integrand in the integral of (5.1)

$$|\sin s - g(s, N) \sin s'| / s > |\sin s| s > \sqrt{3}/2s.$$

Thus the amount contributed to the integral I_δ by one of these half-period intervals, of type $(a, a+\pi)$ will be greater than $(\sqrt{3}/2)(\log (a+2\pi/3) - \log (a+\pi/3)) \sim (\sqrt{3}/6)(\log (a+\pi) - \log a)$, and the total contribution of the interval (s_1, s_2) to I_δ will be greater than $(\sqrt{3}/6) \log (s_2/s_1)$. Hence, finally, if N is sufficiently large,

$$I_\delta > (\sqrt{3}/6)\theta(N) \int_{aN}^{bN} ds/s = (\sqrt{3}/6)\theta(N) \log (b_N/a_N).$$

But $\log (b_N/a_N) \sim (3/4 - 2/3) \log N = (1/12) \log N$, and therefore if

$$\theta(N) \neq o(\log N)^{-1} \qquad (N \to \infty)$$
$$\psi(\delta) \neq o(1/|\log \delta|) \qquad (\delta \to 0)$$

it follows that I_δ is not $o(1)$. In other words, we have proved that:

If $\psi(t)$ is not $o(1/|\log t|)$ then the function

$$\phi(t) = \psi(t) \sin (1/t)$$

does not satisfy the condition (L).

In the following paragraphs it will become clear that *this function does satisfy conditions* (a), (A_1). We begin by proving that the function $\sin (1/t)$ satisfies these conditions.

In order to prove that the function $\sin (1/t)$ satisfies condition (a) we consider the integral:

$$I(\lambda, \mu) = 2 \int_\lambda^\mu \sin (1/t) \sin nt dt/t \qquad (0 < \lambda < \mu)$$
$$= \int_\lambda^\mu \cos (nt - 1/t) dt/t - \int_\lambda^\mu \cos (nt + 1/t) dt/t$$
$$= I_1(\lambda, \mu) - I_2(\lambda, \mu).$$

We set $q=\log(t\sqrt{n})$, $\alpha=\log(\lambda\sqrt{n})$, $\beta=\log(\mu/\sqrt{n})$, and obtain

$$I_1(\lambda,\mu)=\int_\alpha^\beta \cos(2\sqrt{n}\sinh q)dq = J_1(\beta)-J_1(\alpha)$$

$$I_2(\lambda,\mu)=\int_\alpha^\beta \cos(2\sqrt{n}\cosh q)dq = J_2(\beta)-J_2(\alpha)$$

where α, β are arbitrary, $-\infty<\alpha<\beta<\infty$. Now, since the functions cos, cosh q, are even and the function sinh q odd we may take $\alpha=0$, $\beta>0$, in both integrals. Thus

$$J_1(\beta)=\int_0^\beta \cos 2(\sqrt{n}\sinh q)dq$$

$$=(1/2\sqrt{n})\int_0^\beta \operatorname{sech} q\, d(\sin(2\sqrt{n}\sinh q))$$

and since sech q is decreasing, we have sech $0=1$ and

$$J_1(\beta)=(1/2\sqrt{n})\int_0^\gamma d(\sin(2\sqrt{n}\sinh q)$$

$$=O(1/\sqrt{n})$$

Similarly, taking $\alpha=0$, we consider

$$J_2(\beta)=\int_0^\beta \cos(2\sqrt{n}\cosh q)dq.$$

But here we have to consider two cases. First suppose that $\beta\leqq\sinh^{-1}(n^{-1/4})$. Then

$$J_2(\beta)\leqq\sinh^{-1}(n^{-1/4})=O(n^{-1/4}).$$

Second, if $\beta>\sinh^{-1}(n^{-1/4})=b$ we have to consider only

$$J_2(\beta)-J_2(b)=\int_b^\beta \cos(2\sqrt{n}\cosh q)dq.$$

$$=(1/2\sqrt{n})\int_b^\beta \operatorname{csch} q\, d(\sin(2\sqrt{n}\cosh q))$$

and since csch q is decreasing it follows, by the Mean Value Theorem, that the last integral can be expressed as

$$(1/2\sqrt{n})\ \text{csch}\ b \int_b^c d(\sin\ (2\sqrt{n}\ \cosh\ q))$$

$$= (1/2\sqrt{n})n^{1/4} \int_b^c d(\sin\ (2\sqrt{n}\ \cosh\ q)) = O(n^{-1/4}).$$

We have therefore proved that $I(\lambda, \mu) = O(n^{-1/4})$ uniformly for all $\lambda, \mu; 0 < \lambda < \mu$.

With obvious modifications the same sort of proof can be applied to show that the integral

$$\int_\lambda^\mu \sin\ (1/t)\ \cos\ ntdt/t = O(n^{-1/4}).$$

We have therefore proved that the function $\sin\ (1/t)$ satisfies condition (a). It is clear that the same is true of the function $\cos\ (1/t)$.

It is easily seen that these functions satisfy the fundamental condition (A_1), for since, for example,

$$d(t^2\ \cos\ (1/t)) = \sin\ (1/t) + 2t\ \cos\ (1/t)$$

we have

$$\int_0^t \sin\ (1/t)dt = t^2\ \cos\ (1/t) - \int_0^t 2t\ \cos\ (1/t)dt = O(t^2).$$

It is worth while to observe that, by means of the examples $\sin\ (1/x)$ and $\cos\ (1/x)$, it is easy to construct a fairly extensive class of functions of oscillatory character which satisfy condition (a) but not (L). We proceed as follows:

Let $\psi(x)$, $\psi_2(x)$ be bounded functions, each of which has total variation in the interval $[\delta, \pi]$ of the order $O(\tau(\delta))$, where $\tau(\delta)$ is a function which may become positively infinite as $\delta \to \infty$; then the product $\psi_1(x) \cdot \psi_2(x)$ has the same property, since

$$\psi_1(x_2)\psi_2(x_2) - \psi_1(x_1)\psi_2(x_1)$$

$$= \psi_1(x_2)[\psi_2(x_2) - \psi_2(x_1)] + \psi_2(x_1)[\psi_1(x_2) - \psi_1(x_1)].$$

Consider now a function of the type:

$$\phi(x) = \psi(x) \sin (1/x + \theta(x))$$
$$= \psi(x) \cos \theta(x) \sin (1/x) + \psi(x) \sin \theta(x) \cos (1/x)$$

where $\psi(x)$, $\cos \theta(x)$, $\sin \theta(x)$, have the property required of $\psi_1(x)$, $\psi_2(x)$, which will certainly be the case if the total variations of $\psi(x)$, $\theta(x)$, in the interval $[\delta, \pi]$ are $o(\tau(\delta))$ and $\psi(x)$ alone is bounded; for the functions $\sin \theta$, $\cos \theta$ are bounded and $|\sin \theta(x_2) - \sin \theta(x_1)| \leqq |\theta(x_2) - \theta(x_1)|$. This being so, it will follow that the functions $\psi_3(x) = \psi(x) \sin \theta(x)$, $\psi_4(x) = \psi(x) \cos \theta(x)$ will have the required property.

We can now prove that the function $\phi(x)$, thus defined, satisfies the conditions (A_1), (a) provided that (i) the total variations of $\psi(x)$, $\theta(x)$, in the interval $[\delta, \pi]$ are $o(\delta^{-1/4})$ and, (ii) $\psi(x)$ is bounded.

It is easy to prove that $\phi(x)$ satisfies condition (A_1). In fact we prove a stronger result:

If $\psi(x)$ is such that its total variation in the interval $[\delta, \pi]$ is $o(\delta^{-1})$ then the functions $\psi(x) \sin (1/x)$, $\psi(x) \cos (1/x)$, satisfy condition (A_1).

We give the proof for the first function only since the method applies equally well to the second.

Suppose first that $\psi(x)$ is positive and monotonically decreasing, with $\psi(x) = o(x^{-1})$. Since

$$d(x^2 \cos (1/x)) = \sin (1/x)dx + 2x \cos (1/x)$$

it follows that

$$\int_{+0}^{h} \psi(x) \sin (1/x)dx = \int_0^h \psi(x)d(x^2 \cos (1/x))$$
$$- 2\int_0^h \psi(x) \cos (1/x)xdx$$
$$= h^2 \cos (1/h)\psi(h) - \int_0^h x^2 \cos (1/x)d\psi(x)$$
$$- 2\int_0^h \psi(x) \cos (1/x)xdx.$$

Here the first term is $o(h)$. The second term is less, in absolute value, than the positive quantity:

$$-\int_{+0}^{h} x^2 d\psi(x) = -h^2\psi(h) + 2\int_{+0}^{h} \psi(x)\,x\,dx$$

$$\leqq 2\int_{+0}^{h} \psi(x)\,x\,dx.$$

The third term is also less, in absolute value, than

$$2\int_{+0}^{h} \psi(x)\,x\,dx$$

and, since $\psi(x) = o(x^{-1})$, this integral is $o(h)$. Having proved that each of the three terms in question is $o(h)$, we have proved, since the same method applies to the function $\psi(x)\cos(1/x)$, that $\psi(x)\sin(1/x)$, $\psi(x)\cos(1/x)$, satisfy condition (A_1), *if* $\psi(x)$ *is monotone decreasing*. But, by hypothesis, $\psi(x)$ can be expressed as the difference of two such functions each of which is $o(x^{-1})$. Thus the proof is complete. We can now state a theorem which exhibits a fairly general class of functions satisfying the conditions (A_1), (a), but not L, as our first example shows.

THEOREM 3. *If the total variation of each of the functions* $\psi(t)$, $\theta(t)$, *in the interval* $[\delta, \pi]$ *is* $o(\delta^{-1/4})$, *and if the function* $\psi(t)$ *is bounded in* $[0, \pi]$, *then the function*:

$$\phi(t) = \psi(t)\sin(1/t + \theta(t))$$

$$= \psi_1(t)\sin(1/t) + \psi_2(t)\cos(1/t)$$

satisfies conditions (A_1) *and* (a).

The proof of this theorem comes immediately from an application of the Second Mean Value Theorem to integrals of the type:

$$\int_{\lambda}^{\mu} \psi_j(t)\sin(1/t)\cos ntdt/t \qquad (\pi/n < \lambda < \mu \leqq \pi)$$

$$= \psi_j(\lambda) \int_\lambda^m \sin(1/t) \cos nt\, dt/t \qquad (\lambda \leqq m \leqq \mu \quad (j = 1, 2)$$

and those obtained by replacing $\sin(1/t)$ by $\cos(1/t)$ and $\cos nt$ by $\sin nt$. In the case above the integral is $o(1)$ uniformly for all $\lambda, \mu, [\pi/n < \lambda < \mu \leqq \pi]$ since $\psi_j(t)$ may be regarded as a function which is positive, monotone decreasing and of the order $o(t^{-1/4})$, so that the integral is $o(n^{1/4}) \cdot O(n^{-1/4}) = o(1)$ as $n \to \infty$.

6. $\mathcal{F}$-Regularity. The equivalence of the conditions (A′), (B′) provides another interesting criterion for the simultaneous convergence of the series $\mathfrak{S}(f)$ and $\overline{\mathfrak{S}}(f)$ at the point x_0. For this we find it convenient to define properties relating to the behavior of the function $f(x)$ in the right-hand and left-hand neighborhoods respectively of the point x_0.

We consider the Fourier series of the periodic function $f_r(x_0, x)$ defined as follows:

$$f_r(x_0, x) = f(x_0), \qquad y_0 - \pi < x \leqq x_0$$
$$= f(x), \qquad x_0 \leqq x \leqq x_0 + \pi.$$

Let

$$\phi_r(t) = f(x_0 + t) - f(x_0) \qquad (0 \leqq t \leqq \pi)$$
$$= f_r(x_0, x_0 + t) + f_r(x_0, x_0 - t) - 2f_r(x_0, x_0)$$
$$= f_r(x_0, x_0 + t) - f_r(x_0, x_0 - t).$$

Then $\phi_r(t)$ is the function which corresponds to $\phi(t)$ in the formula for $s_n^r - f_r(x_0, x_0)$ and also to $\psi(t)$ in the formula for $\bar{s}_n^r$, where s_n^r, $\bar{s}_n^r$ are the partial sums of $\mathfrak{S}(f_r)$ and $\overline{\mathfrak{S}}(f_r)$. That is:

$$s_n^r - f_r(x_0, x_0) = (1/\pi) \int_0^\pi (\phi_r(t)/t) \sin nt\, dt + o(1)$$

$$\bar{s}_n^r = -(1/\pi) \int_0^\pi (\phi_r(t)/2 \tan(t/2))(1 - \cos nt) dt + o(1).$$

We define $\mathcal{F}$-regularity as follows:

DEFINITION 1. *If* $f(x) \in (L)$ *is of period* 2π, *and is such that the integral*

$$\int_{+0}^{\pi} (f(x_0 + t) - f(x_0))(1/t)dt$$

exists, and if the series $\mathfrak{S}(f_r(x_0, x))$ *and* $\overline{\mathfrak{S}}(f_r(x_0, x))$ *are convergent at* $x = x_0$, *then the function* $f(x)$ *is said to be* $\mathcal{F}$*-regular on the right at* $x = x_0$.

This amounts to the same as saying that $f(x)$ *is* $\mathcal{F}$*-regular on the right if, and only if,* (i) *the integral*

$$\int_{+0}^{\pi} (f(x_0 + t) - f(x_0))dt/t = \int_{+0}^{\pi} (\phi_r(t)/t)dt$$

exists and (ii), *the integral*

$$\int_{+0}^{\pi} (\phi_r(t)/t)e^{int}dt = o(1) \qquad (n \to \infty).$$

Similarly, if we define the periodic function

$$f_l(x_0, x) = f(x), \qquad x_0 - \pi < x \leqq x_0$$
$$= f(x_0), \qquad x_0 \leqq x \leqq x_0 + \pi$$

and

$$\phi_l(t) = f(x_0 - t) - f(x_0) \qquad (t > 0)$$

we have, if $s_n^l(x_0)$ and $\bar{s}_n^l(x_0)$ are partial sums of $\mathfrak{S}(f_l(x_0, x))$ and $\overline{\mathfrak{S}}(f_l(x_0, x))$ respectively,

$$s_n^l(x_0) - f_l(x_0, x_0) = (1/\pi) \int_0^{\pi} (\phi_l(t)/t) \sin ntdt + o(1)$$

$$\bar{s}_n^l(x_0) = (1/\pi) \int_0^{\pi} (\phi_l(t)/2 \tan (t/2))(1 - \cos nt)dt + o(1),$$

and we make the corresponding

DEFINITION 2. *If* $f(x) \in (L)$, *is of period* 2π, *and is such that the integral*

$$\int_{+0}^{\pi}(f(x_0 - t) - f(x_0))(1/t)dt = \int_{+0}^{\pi}(\phi_l(t)/t)dt$$

exists, and if the series $\mathfrak{S}(f_l(x_0, x))$ *and* $\overline{\mathfrak{S}}(f_l(x_0, x))$ *are convergent at* $x = x_0$, *then the function* $f(x)$ *is said to be* $\mathfrak{F}$*-regular on the left at* $x = x_0$.

From these definitions it follows that if the integrals:

$$\int_{+0}^{\pi}(\phi_r(t)/t)dt, \qquad \int_{+0}^{\pi}(\phi_l(t)/t)dt,$$

exist, then *a necessary and sufficient condition in order that* $f(x)$ *be* $\mathfrak{F}$*-regular on the right and on the left is that*

$$\text{(a')} \qquad \int_{+0}^{\pi}(\phi_r(t)/t)e^{int}dt = o(1), \qquad \int_{+0}^{\pi}(\phi_l(t)/t)e^{int}dt = o(1).$$

If now we extend the range of the functions $\phi_r(t)$, $\phi_l(t)$, so as to render them periodic (2π), by means of the definitions

$$\phi_r(t) \equiv 0, \qquad \phi_l(t) \equiv 0 \qquad (\pi < t \leqq 2\pi)$$

$$\phi_r(2k\pi + t) \equiv \phi_r(t), \quad \phi_l(2k\pi + t) \equiv \phi_l(t), \quad (k = \pm 1, \pm 2, \cdots)$$

and then define the functions $F^r(t)$, $F^l(t)$, for all values of t by the formulae:

$$F^r(t) = \int_{+0}^{t}(\phi_r(t)/t)dt, \qquad F^l(t) = \int_{+0}^{t}(\phi_l(t)/t)dt,$$

$$F_1^r(t) = \int_{0}^{t}F^r(t)dt, \qquad F_1^l(t) = \int_{0}^{t}F\ (t)dt$$

it is evident that the condition (a') above is equivalent to the condition:

$$\text{(A')} \qquad \int_{0}^{2\pi}e^{int}dF^r(t) = o(1), \qquad \int_{0}^{2\pi}e^{int}dF^l(t) = o(1);$$

which, by Theorems (A', C'), (B', C'), of §2, is equivalent to the statement that the functions $F^r(t)$, $F^l(t)$, as defined, satisfy condition (B'). We have thus established:

THEOREM 4. *If* $f(x) \in (L)$, *a necessary and sufficient that* $f(x)$ *be* $\mathcal{J}$*-regular on the right and on the left is that both of the functions* $F^r(t)$, $F^l(t)$, *exist, as defined, and satisfy the condition* (B′).

As an immediate consequence of this theorem we can now state, as a criterion for the convergence of both $\mathfrak{S}(f)$ and $\overline{\mathfrak{S}}(f)$:

THEOREM 5. *If* $F^r(t)$, $F^l(t)$, *exist and satisfy the condition* (B′) *then* $\mathfrak{S}(f)$ *and* $\overline{\mathfrak{S}}(f)$ *converge respectively to the sums*:

$$f(x_0) \quad \text{and} \quad -(1/\pi)\int_{+0}^{\pi} \frac{f(x_0+t)-f(x_0-t)}{2\tan(t/2)}\,dt.$$

HUBERT E. BRAY

UNIFORM CONVERGENCE OF FOURIER SERIES

1. Introduction. It is well known that the Fourier series of a continuous function is not necessarily uniformly convergent in an interval of continuity of the sum-function. However, if, in addition to being continuous, the function is also of bounded variation then it is known that the Fourier series is uniformly convergent in the interval of periodicity. This is the so-called Jordan criterion for uniform convergence.

A second criterion for uniform convergence is the Dini-Lipschitz condition. If

$$|f(x+t)-f(x)| < \frac{K}{\left(\log \frac{1}{|t|}\right)^{1+\alpha}} \qquad \alpha, K > 0$$

then the Fourier series of $f(x)$ converges uniformly. This condition may be slightly improved. The essential point to be observed here, as far as the results of this paper are concerned, is that the power of $\log 1/|t|$ exceeds unity.

It was proved by H. E. Bray[1] that continuous functions of *écart fini*[2] have uniformly convergent Fourier series. A function $i(x)$, summable for $0 \leqq x \leqq 2\pi$ is said to be of *écart fini* in this nterval if the integrals

$$\int_a^b f(x)n \sin nx dx \qquad \int_a^b f(x)n \cos nx dx$$

are bounded independently of the integer n and the numbers a, b; where $0 \leqq a < b \leqq 2\pi$. This result contains the above-mentioned Jordan criterion since every function of bounded variation is a function of *écart fini*.

Bray also showed[3] that if the Fourier coefficients of a bounded and measurable function are $O(1/n \log n)$ the function is a function of *écart fini*.

The present paper has a two-fold purpose. First, to define a

[1] Am. Jour. Math., Jan. 1929, p. 149 et seq.
[2] Hadamard. Jour. de Liouville, Ser. 4, v. 8, p. 154.
[3] Comptes Rendus, t 190, p. 1371.

class of functions wider than the class of functions of *écart fini* and to obtain sufficient conditions for the existence of functions of this class. Second, to show that if continuous functions of this new class have moduli of continuity satisfying a slight restriction—less restrictive, in fact, than the Dini-Lipschitz condition —then their Fourier series converge uniformly. These functions do not satisfy the classical convergence criteria. An interesting example of a function of this kind, for which the Dini-Lipschitz criterion is not satisfied, is exhibited.

2. Functions of Class $\phi(n)$. A summable function $f(x)$ of period 2π will be said to be of class $\phi(n)$ in the interval $0 \leqq x \leqq 2\pi$ if

$$\phi(n) \int_a^b f(x+t) \cos ntdt = O(1) \tag{1}$$

uniformly for all x, n, a, b with $b-a \leqq 2\pi$. If, in particular, $\phi(n) = n$, the function $f(x)$ is a function of *écart fini*.

THEOREM 1. *If the Fourier coefficients of a bounded and measurable function $f(x)$ of period 2π are of the order $1/\psi(n)$, where $n \leqq \psi(n) \leqq n \log n$, and if the condition*

$$\frac{d\psi(t)}{dt} > k \frac{\psi(t)}{t} \qquad k > 0^* \tag{2}$$

is satisfied, then $f(x)$ is of class $\phi(n) = \psi(n)/\log n$.

It should be noted that we must have $\phi(n) = O(n)$ except in the trivial case where $f(x)$ is a null function. For suppose that $\phi(n)/n$ were unbounded with $f(x)$ not a null function. We consider the expression

$$\phi(n) \int_0^{\pi/2n} f(x+t) \cos ntdt.$$

An integration by parts gives

$$\int_0^{\pi/2n} f(x+t) \cos ntdt = n \int_0^{\pi/2n} [F(x+t) - F(x)] \sin ntdt$$

where

* This condition assures us that $\psi(t)$ is an increasing function.

$$F(x+t) = F(x) + \int_0^t f(x+y)dy.$$

If we suppose that $f(x)$ is not a null function (we may assume that $f(x)>0$), then for some x we shall have

$$\frac{F(x+t) - F(x)}{t} > q > 0$$

for $|t|$ sufficiently small. Hence

$$n\int_0^{\pi/2n} [F(x+t) - F(x)] \sin ntdt > n\int_0^{\pi/2n} qt \sin ntdt = q/n$$

for n sufficiently large. Therefore

$$\phi(n)\int_0^{\pi/2n} f(x+t)\cos ntdt > \frac{\phi(n)}{n} q. \tag{3}$$

But as $n\to\infty$, $\phi(n)/n\to\infty$. It follows that the expression (3) is unbounded. This is in contradiction with the definition of functions of this class, for $f(x)$ is summable and therefore expression (3) is uniformly bounded.

In order to prove the theorem we make use of two lemmas. The first concerns the expression

$$E(x, m, \delta) = \sum_{r=1}^{m} (-1)^r [F(x+r\delta) - F(x+(r-1)\delta)]$$

with m a positive integer and δ any positive number such that $m\delta \leqq 2\pi$.

LEMMA 1. *If $\phi(n) = O(n)$, and if $f(x)$ is bounded and measurable and is such that $E(x, m, \delta) = O(1/\phi(1/\delta))$, then*

$$\int_a^b f(x+t)\cos ntdt = O(1/\phi(n)).$$

Since $f(x)$ is bounded, a part of the interval of length $O(1/n)$ may be disregarded in considering the boundedness of the expression $|\phi(n)\int_a^b f(x+t)\cos ntdt|$. But $\phi(n) \leqq O(n)$ so we may discard a part of $[a, b]$ of length $O(1/n)$. The integral to be considered is the difference of two integrals of the form

$\int_0^a f(x+t)\cos ntdt$ so it suffices to consider this integral. We have, after letting $a=m\pi/n$ and integrating by parts,

$$\int_0^{m\pi/n} f(x+t)\cos ntdt = (-1)^m F\left(x+m\frac{\pi}{n}\right) - F(x) + \int_0^{m\pi/n} F(x+t)n\sin ntdt.$$

But

$$\int_0^{m\pi/n} F(x+t)n\sin ntdt$$
$$= \frac{1}{2}\int_0^{\pi/n} [F(x+t) + (-1)^{m-1}F(x+t+(m-1)\pi/n)]n\sin ntdt$$
$$+ \frac{1}{2}\int_0^{\pi/n} E(x+t, m-1, \pi/n)n\sin ntdt.$$

Since $E=O(1/\phi(n))$, the last integral is $O(1/\phi(n))$. We therefore have

$$\int_0^{m\pi/n} f(x+t)\cos ntdt$$
$$= \frac{1}{2}\int_0^{\pi/n} [F(x+t) + (-1)^{m-1}F(x+t+(m-1)\pi/n)]n\sin ntdt$$
$$- F(x) + (-1)^m F(x+m\pi/n) + O(1/\phi(n)).$$

This can be written as

$$\int_0^{m\pi/n} f(x+t)\cos ntdt$$
$$= \frac{1}{2}\int_0^{\pi/n} [F(x+t) - F(x)]n\sin ntdt$$
$$+ \frac{1}{2}(-1)^{m-1}\int_0^{\pi/n}\left[F\left(x+t+(m-1)\frac{\pi}{n}\right) - F\left(x+\frac{m\pi}{n}\right)\right]n\sin ntdt + O(1/\phi(n)). \tag{4}$$

But

$$F(x+t) - F(x) = O\left(1/\phi\left(\frac{1}{t}\right)\right) = O(1/\phi(n))$$

Since $O \leqq t \leqq \pi/n$. We may also write

$$\begin{aligned} F(x + t + (m-1)\pi/n) &- F(x + \pi m/n) \\ &= F(x + (m-1)\pi/n + t) - F(x + (m-1)\pi/n) \\ &\quad - [F(x + (m-1)\pi/n + \pi/n) - F(x + (m-1)\pi/n)], \end{aligned}$$

and thus

$$F(x + t + (m-1)\pi/n) - F(x + m\pi/n) = O(1/\phi(n)).$$

Hence each term in the right member of (4) is of the order $1/\phi(n)$, and the lemma is proved.

The trapezoidal sum is used in the proof of the second lemma. This sum is defined on the interval $a \leqq x \leqq b$ for the function $F(x)$ as follows:

$$\begin{aligned} T(a, b, h) = (h/2)[F(a) + 2F(a+h) + 2F(a+2h) + \cdots \\ + 2F(a + (m-1)h) + F(a + mh)] \end{aligned}$$

where $a + mh = b$.

LEMMA 2. *If* $T(a, b, h) - \int_a^b F(x)dx = O(h/\phi(1/h))$ *uniformly in* $a \leqq x \leqq b$, *then* $E(x, m, h) = O(1/\phi(1/h))$.

We have

$$T(a, b, h) - \int_a^b F(x)dx = O(h/\phi(1/h))$$

$$T(a, b, h/2) - \int_a^b F(x)dx = O([h/2]/\phi(2/h)) = O(h/\phi(1/h)).$$

Thus $T(a, b, h) - T(a, b, h/2) = O(h/\phi(1/h))$, and

$$\begin{aligned} F(a) - 2F(a + h/2) + \cdots + (-1)^{2m}F(a + mh) \\ = O(1/\phi(1/h)). \end{aligned}$$

This is the desired result, and the lemma is proved.

We are now able to prove Theorem 1. From condition (2) we obtain the condition

$$\int_n^\infty \frac{dt}{t\psi(t)} = O(1/\psi(n)). \tag{5}$$

For since $\psi'(t)/\psi(t) > k/t$ we have

$$\int_n^\infty \frac{dt}{t\psi(t)} < \frac{1}{k}\int_n^\infty \frac{\psi'(t)}{(\psi(t))^2}\,dt = \frac{1}{k\psi(n)}.$$

We may suppose that, almost anywhere,

$$f(x) = \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

since it is clear that when (1) is satisfied for $f(x)$ it is satisfied for $f(x) - a_0/2$. There is no loss of generality in supposing also that

$$F(x) = \sum_{r=1}^{\infty} \frac{a_r \sin rx - b_r \cos rx}{r}.$$

In order to prove the theorem it will suffice to show that

$$T(a, b, h) - \int_a^b F(x)dx = O\left(\frac{h \log (1/h)}{\psi(1/h)}\right).$$

For the sake of brevity we shall write $\phi(t) = \psi(t)/\log t$. We thus wish to show that

$$T(a, b, h) - \int_a^b F(x)dx = O(h/\phi(1/h)).$$

If we set $b = a + mh$ with $0 < b - a \leqq 2\pi$, we obtain

$$T(a, b, h) = \frac{h}{2}\left[F(a) + 2F(a+h) + \cdots + F(a+mh)\right]$$

$$= \frac{h}{2}\left[\sum_{i=0}^{m-1} F(a+ih) + \sum_{i=1}^{m} F(a+ih)\right] \tag{6}$$

$$= \frac{h}{2} \sum_{r=1}^{\infty} \left[\sum_{i=0}^{m-1} + \sum_{1}^{m} \frac{a_r \sin r(a+ih) - b_r \cos r(a+ih)}{r} \right].$$

But

$$\sum_{i=0}^{m-1} + \sum_{i=1}^{m} \frac{a_r \sin r(a+ih)}{r} = \frac{a_r}{r} (\cos ra - \cos r(a+mh)) \operatorname{ctn} \frac{rh}{2}$$

$$\sum_{i=0}^{m-1} + \sum_{i=1}^{m} \frac{b_r \cos r(a+ih)}{r} = \frac{b_r}{r} (-\sin ra + \sin r(a+mh)) \operatorname{ctn} \frac{rh}{2}.$$

Thus we obtain

$$(7) \qquad T(a, b, h) = \frac{h}{2} \sum_{r=1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r} \operatorname{ctn} \frac{rh}{2} + \sigma$$

where $A_n(x) = a_n \cos nx + b_n \sin nx$. The accent indicates that in the summation r runs through all positive integer values except those for which $rh/2$ is a multiple of π. These will be considered separately, their sum being represented by σ.

We also have

$$\int_a^{a+mh} F(x)dx = -\sum_{r=1}^{\infty} \frac{A_r(x)}{r^2} \Big]_a^{a+mh}$$

$$= -\sum_{r=1}^{\infty}{}' \frac{A_r(a+mh) - A_r(a)}{r^2}$$

with the same interpretation of the prime over the summation

$$(8) \qquad \begin{aligned} T(a, b, h) - \int_a^b F(x)dx &= \frac{h}{2} \sum_{1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r^2} \operatorname{ctn} \frac{rh}{2} + \sigma \\ &\quad - \sum_{1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r^2} \\ &= \sum_{r=1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r^2} \left(\frac{rh}{2} \operatorname{ctn} \frac{rh}{2} - 1 \right) + \sigma. \end{aligned}$$

We wish to show that (8) is $O(h/\phi(1/h))$, that is

$$(9)\qquad \sum_{r=1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r^2 h}\phi(1/h)\left(\frac{rh}{2}\operatorname{ctn}\frac{rh}{2} - 1\right) + \sigma\phi(1/h)/h = O(1).$$

We consider first the terms of σ of (7) for which $rh/2 = k\pi$, $k = 1, 2, 3, \cdots$. We replace h by π/n and consider the terms for which $r = 2kn$. If the right number of (6) is divided by $h/\phi(1/h) = \pi/n\phi(n/\pi)$ we obtain

$$\frac{\pi}{2n}\sum_{k=1}^{\infty}\left\{\sum_{i=0}^{m-1} + \sum_{1}^{m}\left[a_{2kn}\sin 2kn(a+ih) - b_{2kn}\cos 2kn(a+ih)\right]\right\}\frac{n\phi(n/\pi)}{\pi}$$

which is equal to

$$(10)\qquad \frac{\phi\left(\frac{n}{\pi}\right)}{2}\sum_{k=1}^{\infty}\frac{a_{2kn}\sin 2akn - b_{2kn}\cos 2akn}{2kn}\,2m.$$

Since $mh = m\pi/n \leqq 2\pi$ we have $m = O(n)$. The expression (10) does not exceed in absolute value a quantity of the order

$$(11)\qquad m\phi(n)\sum_{k=1}^{\infty}\frac{1}{kn\psi(2kn)}.$$

But

$$\sum_{k=1}^{\infty}\frac{1}{k\psi(2kn)} \leqq \int_1^{\infty}\frac{dt}{t\psi(nt)} = \int_n^{\infty}\frac{dt}{t\psi(t)} = O(1/\psi(n)) \qquad \text{from (5).}$$

It follows that (11) is of order $\phi(n)/\psi(n) = 1/\log n$ which is $o(1)$. This is a sharper result than we need.

If we denote the expression (9) by $P(a, b, h)$ and replace h by π/n we may write

$$P(a, b, h)$$

$$= \sum_{r=1}^{[n]} \frac{A_r(a)-A_r(a+m\pi/n)}{r^2\pi}\,\phi\left(\frac{n}{\pi}\right)\left(\frac{r\pi}{2n}\operatorname{ctn}\frac{r\pi}{2n}-1\right)n$$

$$(12)\qquad + \sum_{[n]+1}^{\infty}{}' \frac{A_r(a)-A_r\left(a+m\dfrac{\pi}{n}\right)}{2r}\,\phi\left(\frac{n}{\pi}\right)\operatorname{ctn}\frac{r\pi}{2n}$$

$$- \sum_{[n]+1}^{\infty} \frac{A_r(a)-A_r\left(a+m\dfrac{\pi}{n}\right)}{r^2\pi}\,n\phi\left(\frac{n}{\pi}\right) + 0(1).$$

The third sum here is in absolute value not greater than

$$\sum_{r=[n]+1}^{\infty}\left|\frac{A_r(a)-A_r\left(a+m\dfrac{\pi}{n}\right)}{r^2\pi}\,n\phi\left(\frac{n}{\pi}\right)\right| < \sum_{[n]+1}^{\infty}\left|\frac{K\phi\left(\dfrac{n}{\pi}\right)n}{r^2\pi\psi(r)}\right|$$

where K is some positive constant. The above expression does not exceed a quantity of the order

$$n\phi(n)\sum_{[n]+1}^{\infty}\left|\frac{1}{r^2\psi(r)}\right| < \frac{\phi(n)}{\psi(n)} = o(1), \qquad \text{by (5).}$$

The second sum of (12) does not exceed in absolute value a quantity of the order

$$(13)\qquad \sum_{r=[n]+1}^{\infty}{}'\left|\frac{\phi(n)\operatorname{ctn} r\pi/2n}{r\psi(r)}\right| = \sum_{[n]+1}^{2[n]-1} + \sum_{2[n]+1}^{\infty}{}'\left|\frac{\phi(n)\operatorname{ctn} r\pi/2n}{r\psi(r)}\right|.$$

The first sum of (13) does not exceed

$$\frac{\phi(n)}{n\psi(n)}\sum_{[n]+1}^{2[n]-1}\left|\operatorname{ctn}\frac{r\pi}{2n}\right| = \frac{1}{n\log n}\sum_{1}^{n-1}\left|\operatorname{ctn}\left(\frac{r\pi}{2n}+\frac{\pi}{2}\right)\right|$$

$$= \frac{1}{n\log n}\sum_{1}^{n-1}\left|\tan\frac{r\pi}{2n}\right|.$$

But

$$\sum_{r=1}^{n-1}\left|\tan\frac{r\pi}{2n}\right| < \int_1^{n-1}\tan\frac{\pi x}{2n}\,dx = -\frac{2n}{\pi}\log\tan\frac{\pi}{2n}.$$

Since $\tan(\pi/2n) = O(1/n)$ it follows that

$$\frac{2n}{\pi} \log \tan(\pi/2n) = O(n \log n).$$

Hence the first sum of (13) has no greater order than

$$[1/(n \log n)]O(n \log n) = O(1).$$

For the second sum in (13) we have

$$(14) \qquad \sum_{r=2n+1}^{\infty}{}' \left| \frac{\phi(n) \operatorname{ctn} \frac{r\pi}{2n}}{r\psi(r)} \right| \leqq \phi(n) \sum_{k=1}^{\infty} \sum_{r=2nk+1}^{2n(k+1)-1} \left| \frac{\operatorname{ctn} \frac{r\pi}{2n}}{r\psi(r)} \right|.$$

The second sum here gives

$$\sum_{r=2nk+1}^{2n(k+1)-1} \left| \frac{\operatorname{ctn} \frac{r\pi}{2n}}{r\psi(r)} \right| < \frac{1}{2nk\psi(2nk)} \sum_{2nk+1}^{2n(k+1)-1} \left| \operatorname{ctn} \frac{r\pi}{2n} \right|$$

$$< \frac{1}{2nk\psi(2nk)} \sum_{r=1}^{2n-1} \left| \operatorname{ctn} \frac{r + 2nk}{2n} \pi \right|$$

$$< \frac{1}{2nk\psi(2nk)} \sum_{r=1}^{2n-1} \left| \operatorname{ctn} \frac{r\pi}{2n} \right|.$$

But

$$\sum_{r=1}^{2n-1} \left| \operatorname{ctn} \frac{r\pi}{2n} \right| = \sum_{r=1}^{n} + \sum_{n+1}^{2n-1} \left| \operatorname{ctn} \frac{r\pi}{2n} \right|$$

and

$$\sum_{r=1}^{n} \left| \operatorname{ctn} \frac{r\pi}{2n} \right| < \int_1^n \operatorname{ctn} \frac{\pi x}{2n} \, dx = \frac{2n}{\pi} \left| \log \sin \frac{\pi}{2n} \right| = O(n \log n).$$

Similarly

$$\sum_{r=n+1}^{2n-1} \left| \operatorname{ctn} \frac{r\pi}{2n} \right| < O(n \log n).$$

It therefore follows that (14) does not exceed a quantity of the order

$$\phi(n)\sum_{k=1}^{\infty}\frac{n\log n}{nk\psi(nk)} \leqq \phi(n)\log n\sum_{k=1}^{\infty}\frac{1}{k\psi(nk)}. \tag{15}$$

We have already seen that the last sum in (15) is not greater than a quantity of the order $1/\psi(n)$. It follows that (14) is of the order $\phi(n)\log n/\psi(n)=1$.

We have shown thus far that the second and third sums of (12) are $O(1)$. We must now show that

$$\sum_{r=1}^{n}\frac{A_r(a)-A_r(a+m\pi/n)}{r^2\pi}\,n\phi\left(\frac{n}{\pi}\right)\left(\frac{r\pi}{2n}\operatorname{ctn}\frac{r\pi}{2n}-1\right)=O(1).$$

The expression

$$\frac{1-\dfrac{r\pi}{2n}\operatorname{ctn}\dfrac{r\pi}{2n}}{r^2\pi}\,n\phi\left(\frac{n}{\pi}\right) \tag{16}$$

is bounded as $n\to\infty$ for fixed r. This is easily seen since from a well known formula*

$$\frac{1-x\operatorname{ctn}x}{x^2}=\frac{1}{3}+\frac{x^2}{45}+\frac{2x^4}{945}+\frac{x^6}{4725}+\cdots.$$

Moreover, (16) is an increasing function of r for each fixed n. Its greatest value occurs therefore when $r=n$; namely

$$\frac{\phi\left(\dfrac{n}{\pi}\right)}{n\pi}\leqq K/\pi^2,$$

where K is a positive constant.

If we use the second mean value theorems for sums, we obtain

$$\frac{K}{\pi^2}\sum_{k'\leqq n}^{n}\left[A_r\left(a+m\frac{\pi}{n}\right)-A_r(a)\right]$$

* B. O. Peirce, Tables.

$$\leqq \sum_1^n \left[A_r\left(a + m\frac{\pi}{n}\right) - A_r(a) \right] \frac{1 - \frac{r\pi}{2n} \operatorname{ctn} \frac{r\pi}{2n}}{r^2\pi} n\phi\left(\frac{n}{\pi}\right)$$

$$\leqq \frac{K}{\pi^2} \sum_{k'' \leqq n}^{n} \left[A_r\left(a + m\frac{\pi}{n}\right) - A_r(a) \right].$$

It is thus sufficient to show that

$$\sum_1^n \left[A_r\left(a + m\frac{\pi}{n}\right) - A_r(a) \right]$$

is uniformly bounded for all n—i.e., that $\sum_1^n A_r(x)$ is uniformly bounded.

If $f(x)$ is bounded and measurable, the first Cesàro mean is uniformly bounded for all x and n. For if the nth partial sum of this mean is denoted by σ_n, we have

$$\sigma_n(x) = \frac{1}{n\pi} \int_0^\pi f(x + 2t) \frac{\sin^2 nt}{\sin^2 t}\, dt$$

$$|\sigma_n(x)| \leqq \frac{B}{n\pi} \int_0^\pi \frac{\sin^2 nt}{\sin^2 t}\, dt = B, \quad (|f(x)| \leqq B).$$

From this the result follows. For if $f(x)$ is bounded and has Fourier coefficients $O(1/n)$, the partial sums of the Fourier development are uniformly bounded. In fact

$$s_n - \sigma_n = \frac{A_1 + 2A_2 + \cdots + nA_n}{n+1}$$

where s_n is the nth partial sum of $\sum_1^\infty An(x)$. But $nA_n = O(1)$, so it follows that the partial sums of the series are bounded uniformly. This completes the proof of Theorem 1.

As a particular example we note that the conditions of the theorem are satisfied for the function $\psi(n) = n(\log n)^\alpha$, $0 < \alpha \leqq 1$. For we have

$$\frac{d\psi}{dn} = \frac{\alpha}{(\log n)^{1+\alpha}} + (\log n)^\alpha > (\log n)^\alpha = \frac{\psi(n)}{n}.$$

Since the condition that $\psi(n) \geqq n$ was used in the proof of Theorem 1 only to show that the partial sums of the Fourier series of $f(x)$ were uniformly bounded, we may state the following theorem.

COROLLARY TO THEOREM 1. *If the Fourier coefficients of a bounded and measurable function $f(x)$ of period 2π are of the order $1/\psi(n)$ with the condition $d\psi(t)/dt > k\psi(t)/t$ satisfied for some $k>0$, and if the partial sums of the Fourier series of $f(x)$ are uniformly bounded, then $f(x)$ is of class $\psi(n)/\log n$.*

Following the terminology of Bray,[4] we shall say that a function $f(x)$ *is of class $\phi(n)$ and admits the constant K if K is such that the inequalities*

$$\left| \phi(n) \int_a^b f(x+k) \cos nx dx \right| \leqq K$$

$$\left| \phi(n) \int_a^b f(x+k) \sin nx dx \right| \leqq K$$

are satisfied for all $n \geqq 1$, k arbitrary, and $b-a \leqq 2\pi$.

LEMMA 3. *If $f(x)$ is of class $\phi(n)$, admitting the constant K, and if $|f(x)| \leqq M$, there exist positive constants A and B independent of $f(x)$ such that*

$$|S_n(x)| \leqq M\left(A \log \theta(n) + B \frac{n}{\phi(n)}\right) + \frac{2K}{\theta(n)}$$

where $S_n(x)$ is the nth partial sum of the Fourier series of $f(x)$ and $\theta(n)$ is any monotone increasing function such that $1 \leqq \theta(n) \leqq \phi(n)$.

In order to prove this we consider

$$S_n(x) = \frac{1}{\pi} \int_0^{\pi} [f(x+t)+f(x-t)] \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2}t} dt$$

$$= \frac{1}{\pi} \int_0^{\theta(n)/\phi(n)} + \int_{\theta(n)/\phi(n)}^{\pi} [f(x+t)+f(x-t)] \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2}t} dt.$$

[4] Loc. cit., p. 160.

Hence

$$|S_n(x)| \leqq \frac{M}{\pi}\int_0^{\theta(n)/\phi(n)} \left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\right| dt$$

$$+\frac{1}{\pi}\left|\int_{\theta(n)/\phi(n)}^{\pi} f(x+t)+f(x-t)]\,\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt\right|. \tag{17}$$

The first integral in (17) does not exceed

$$\int_0^{1/\phi(n)} \left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\right| dt + \int_{1/\phi(n)}^{\theta(n)/\phi(n)} \frac{4}{t}\,dt$$

which is not greater than $(2n+1)/\phi(n)+4\log\theta(n)$. Thus the first term of (17), is not greater than

$$\frac{M}{\pi}\left[\frac{2n+1}{\phi(n)}+4\log\theta(n)\right].$$

For the second integral we obtain, by using the second mean value theorem,

$$\int_{\theta/\phi}^{\pi} [f(x+t)+f(x-t)]\,\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

$$= \tfrac{1}{2}\csc\frac{\theta(n)}{2\phi(n)}\int_{\theta(n)/\phi(n)}^{t_n} [f(x+t)+f(x-t)]\sin(n+\tfrac{1}{2})t\,dt$$

where

$$\frac{\theta(n)}{\phi(n)} \leqq t_n \leqq \pi.$$

But

$$\left|\int_a^b f(x+t)\sin nx\,dx\right| \leqq K/\phi(n).$$

Hence

$$\int_{\theta(n)/\phi(n)}^{t_n} [f(x+t)+f(x-t)]\sin(n+\tfrac{1}{2})t\,dt$$

$$\leqq \frac{2K}{\phi(n+\frac{1}{2})} \leqq \frac{2K}{\phi(n)}.$$

Thus the second term of (17) does not exceed

$$\frac{1}{2\pi}\csc\frac{\theta(n)}{2\phi(n)}\frac{2K}{\phi(n)} \leqq \frac{1}{2\pi}\frac{4\phi(n)}{\theta(n)}\frac{2K}{\phi(n)} = \frac{4K}{\pi\theta(n)} \leqq \frac{2K}{\theta(n)}$$

and we get the desired result.

LEMMA 4. *Under the hypotheses of Theorem* 3 *there exist positive constants* A *and* B *independent of* $f(x)$ *such that the remainder* $R_n(x)=f(x)-S_n(x)$ *satisfies the inequality*

$$|R_n| \leqq M\left(A\log\theta(n) + B\frac{n}{\phi(n)}\right) + \frac{2K}{\theta(n)}.$$

For

$$\begin{aligned}|R_n| &= |S_n - f| \leqq |S_n| + |f| \\ &\leqq M\left[A\log\theta(n) + B'\frac{n}{\phi(n)}\right] + \frac{2K}{\theta(n)} + M; \\ |R_n| &\leqq M\left[A\log\theta(n) + B'\frac{n}{\phi(n)} + 1\right] + \frac{2K}{\theta(n)} \\ &\leqq M\left[A\log\theta(n) + B\frac{n}{\phi(n)}\right] + \frac{2K}{\theta(n)}\end{aligned}$$

since

$$B'\frac{n}{\phi(n)} + 1 \leqq (B'+1)\frac{n}{\phi(n)} = B\frac{n}{\phi(n)}.$$

LEMMA 5. If $f(x)$ has first and second order derivatives $f'(x)$, $f''(x)$, and if $f''(x)$ is of class $\phi(n)$ admitting the constant K'', with $|f''(x)| \leqq M''$; then if $\phi'(n) < B'$ there exist positive constants A and B such that

$$|R_n| \leqq \frac{M''}{n^2}\left[A\log\theta(n) + B\frac{n}{\phi(n)}\right] + \frac{4K''}{n^2\theta(n)}$$

where $\theta(n)$ is any monotone increasing function such that

$$1 \leqq \theta(n) \leqq \phi(n); \qquad 1 \leqq \frac{\theta(n+1)}{\theta(n)} \leqq \frac{\phi(n+1)}{\phi(n)}.$$

We let $S_n(x) = \sum_1^n A_k(x)$ and let the Fourier partial sum of $f''(x)$ be

$$\sigma_n = \sum_1^n B_k(x) = -\sum_1^n k^2(a_k \cos kx + b_k \sin kx)$$

$$= -\sum_1^n k^2 A_k(x).$$

We thus obtain

$$R_n = \sum_{n+1}^{\infty} A_k(x) = -\sum_{n+1}^{\infty} \frac{B_k}{k^2} = \sum_{n+1}^{\infty} \frac{1}{k^2}(\sigma_k - \sigma_{k-1}).$$

Hence

$$R_n = \frac{\sigma_n}{(n+1)^2} - \sum_{n+1}^{\infty} \sigma_k \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right).$$

But

$$|\sigma_k| \leqq M''\left(A \log \theta(k) + B\frac{k}{\phi(k)}\right) + \frac{2K''}{\theta(k)}$$

from Lemma 3. Hence we have

$$|R_n| \leqq \frac{M''}{(n+1)^2}\left[A \log \theta(n) + B\frac{n}{\phi(n)}\right] + \frac{2K''}{\theta(n)(n+1)^2}$$

$$+ 2K''\sum_{n+1}^{\infty} \frac{k}{\phi(k)}\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) \tag{18}$$

$$+ AM''\sum_{n+1}^{\infty} \log \theta(k)\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right).$$

The fourth term of (18) does not exceed

$$\frac{2K''}{\theta(n+1)}\sum_{n+1}^{\infty}\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) \leqq \frac{2K''}{\theta(n+1)}\,\frac{1}{(n+1)^2}.$$

The fifth term is equal to

$$(19)\quad AM''\left[\frac{\log\theta(n+1)}{(n+1)^2}+\sum_{n+1}^{\infty}\left\{\left(\log\frac{\theta(k+1)}{\theta(k)}\right)\frac{1}{(k+1)^2}\right\}\right].$$

The second term of (19) is not greater than

$$\sum_{n+1}^{\infty}\frac{1}{(k+1)^2}\log\frac{\theta(k+1)}{\theta(k)}\leqq\sum_{n+1}^{\infty}\frac{B'}{\phi(k)}\frac{1}{(k+1)^2}$$
$$\leqq\frac{B'}{\phi(n)}\sum_{n+1}^{\infty}\frac{1}{(k+1)^2}\leqq\frac{1}{n^2}B'\frac{n}{\phi(n)}$$

since

$$\log\frac{\phi(k+1)}{\phi(k)}\leqq\frac{B'}{\phi(k)}.$$

This follows at once from the law of the mean and one of the hypotheses of the theorem.

Thus the fifth term of (18) does not exceed

$$(20)\qquad AM''\left(\frac{\log\theta(n+1)}{(n+1)^2}+\frac{1}{n^2}B'\frac{n}{\phi(n)}\right).$$

Since $\phi'(n)\leqq B$ it follows that

$$\phi(n+1)\leqq B'+\phi(n)\leqq 2\phi(n)$$

for n sufficiently large. Then $\theta(n+1)/\theta(n)\leqq 2$ and $\theta(n+1)\leqq 2\theta(n)$. Hence

$$\log\theta(n+1)\leqq\log\theta(n)+\log 2\leqq 2\log\theta(n)$$

and the expression (20) is not greater than

$$\frac{AM''}{n^2}\left(2\log\theta(n)+\frac{1}{n^2}B\frac{n}{\phi(n)}\right).$$

The third term of (19) does not exceed

$$\frac{BM''}{\phi(n)}\sum_{n+1}^{\infty}k\left(\frac{2k+1}{k(k+1)^2}\right)\leqq\frac{2BM''}{\phi(n)}\sum_{n+1}^{\infty}\frac{1}{k^2}$$
$$\leqq\frac{2BM''}{n\phi(n)}\leqq\frac{M''}{n^2}2B\frac{n}{\phi(n)}.$$

By combining the above results we get

$$|R_n| \leqq \frac{M}{n^2}\left(3A \log \theta(n) + 3B \frac{n}{\phi(n)}\right) + \frac{4K''}{n^2\theta(n)}$$

which has the desired form.

THEOREM 2. *If $f(x)$ is of class $\phi(n)$ with $\phi'(n) < B'$ and if $f(x)$ admits the constant K and is continuous with modulus of continuity $\omega(\delta)$, then there exist positive constants A, B, and C independent of $f(x)$ such that*

$$|R_n| \leqq \omega\left(\frac{1}{n}\right)\left[A \log \theta(n) + B \frac{n}{\phi(n)}\right] + \frac{CK}{\theta(n)} \tag{21}$$

where $\theta(n)$ is monotone increasing and such that

$$1 \leqq \theta(n) \leqq \phi(n); \qquad 1 \leqq \frac{\theta(n+1)}{\theta(n)} \leqq \frac{\phi(n+1)}{\phi(n)}.$$

In order to prove this theorem we employ the averaging function[5]

$$f_\mu(x) = \frac{1}{\mu^2}\int_0^\mu dt \int_0^\mu f(x+t+t')dt'.$$

This function has the following properties, as may be easily verified:

(i) If $f(x)$ is continuous,

$$\frac{d}{dx} f_\mu(x) = \frac{1}{\mu^2}\int_0^\mu [f(x+\mu+t) - f(x+t)]dt.$$

(ii) If $f(x)$ is continuous,

$$\frac{d^2}{dx^2} f_\mu(x) = \frac{1}{\mu^2}[f(x+2\mu) - 2f(x+\mu) + f(x)].$$

(iii) If $f(x)$ is continuous, of class $\phi(n)$, admits the constant K, and has modulus of continuity $\omega(\delta)$, then $f_\mu(x)$ is of class $\phi(n)$ and admits the constant K.

[5] See Bray, Bull. Am. Math. Soc. Vol. 29, No. 6.

(iv) Under the conditions of (iii) $f_\mu''(x)$ is of class $\phi(n)$ and admits the constant $4K/\mu^2$.

(v) Under the conditions of (iii)

$$|f_\mu''(x)| \leqq \frac{2\omega(\mu)}{\mu^2}.$$

(vi) Under the conditions of (iii)

$$|f_\mu(x) - f(x)| \leqq \omega(2\mu).$$

In order to prove the theorem we write

$$f(x) = (f(x) - f_\mu(x)) + f_\mu(x).$$

We then have $R_n = R_n' + R''$ where R_n, R_n', and R'' are the Fourier remainders of order n of $f(x)$, $f(x) - f_\mu(x)$, and $f_\mu(x)$, respectively. It is clear from (iii) that $f(x) - f_\mu(x)$ is of class $\phi(n)$ and admits the constant $2K$. From Lemma 4, using the constants of Lemma 5, we have

$$|R_n'| \leqq \omega(2\mu)\left(A \log \theta(n) + B\frac{n}{\phi(n)}\right) + \frac{4K}{\theta(n)}.$$

Since $f_\mu''(x)$ is of class $\phi(n)$ and admits the constant $4K/\mu^2$ we have from Lemma 5

$$|R_n''| \leqq \frac{2\omega(\mu)}{\mu^2 n^2}\left[A \log \theta(n) + B\frac{n}{\phi(n)}\right] + \frac{16K}{n^2\mu^2\theta(n)}.$$

If we put $\mu = 1/2n$, we then have

$$\begin{aligned} |R_n| &= |R_n'| + |R_n''| \\ &\leqq 9\omega\left(\frac{1}{n}\right)\left[A \log \theta(n) + B\frac{n}{\phi(n)}\right] + \frac{68K}{\theta(n)}, \end{aligned}$$

since $\omega(1/2n) \leqq \omega(1/n)$.

Thus there are positive constants A, B, and C such that

$$|R_n| \leqq \omega\left(\frac{1}{n}\right)\left(A \log \theta(n) + B\frac{n}{\phi(n)}\right) + \frac{CK}{\theta(n)},$$

and the theorem is proved.

From this last theorem we see that if $\phi(n)$ is such that $\omega(1/n)n/\phi(n)\to 0$ as $n\to\infty$ and if $\theta(n)$ is so chosen that $\omega(1/n)\log\theta(n)\to 0$ while $\theta(n)\to\infty$, then the Fourier series of $f(x)$ converges uniformly to $f(x)$.

3. EXAMPLE. *A Function of Class* $\phi(n)=n/(\log n)^{1/3}$ *having a Uniformly Convergent Fourier Series*:

The function we shall consider is defined as follows. We let

$$f(x) = k(x)\sum_{i=1}^{\infty}\lambda_i(x) \qquad \text{for } x\neq 0$$

$$= 0 \qquad \text{for } x = 0$$

where

$$\lambda_i(x) = \sin n_i x \qquad \text{for } \alpha_i \leqq x \leqq \beta_i$$

$$= 0 \qquad \text{otherwise.}$$

We set

$$\mu(x) = 1\Big/\left(\log\frac{1}{x}\right)^{2/3}, \qquad 0 \leqq x \leqq \pi/4$$

$$= \mu(\pi/4), \qquad \pi/4 \leqq x,$$

and let

$$k(x) = \mu(\alpha_i) \qquad \text{for } \alpha_i \leqq x \leqq \beta_i$$

$$= 0 \qquad \text{otherwise,}$$

$n_i=2^i2^{2^i}$, $\alpha_i=4\pi/2^{2^i}$. β_i is a number such that $\beta_i>\alpha_i$ and $\beta_i\sim\alpha_i$. For convenience we take $\beta_i=2\alpha_i$.

By the notation $\alpha\sim\beta$ we shall mean that α/β is bounded from zero and infinity. We say that α and β have the same order, and we shall also write $\alpha/\beta\sim 1$.

It is clear that the function $f(x)$ defined above is continuous in $0\leqq x\leqq 2\pi$. The graph of the function is illustrated in the accompanying figure.

If the function $\phi(n)$ is the function $n/(\log n)^{1/3}$, we shall show first that $f(x)$ is at least of this class. We have

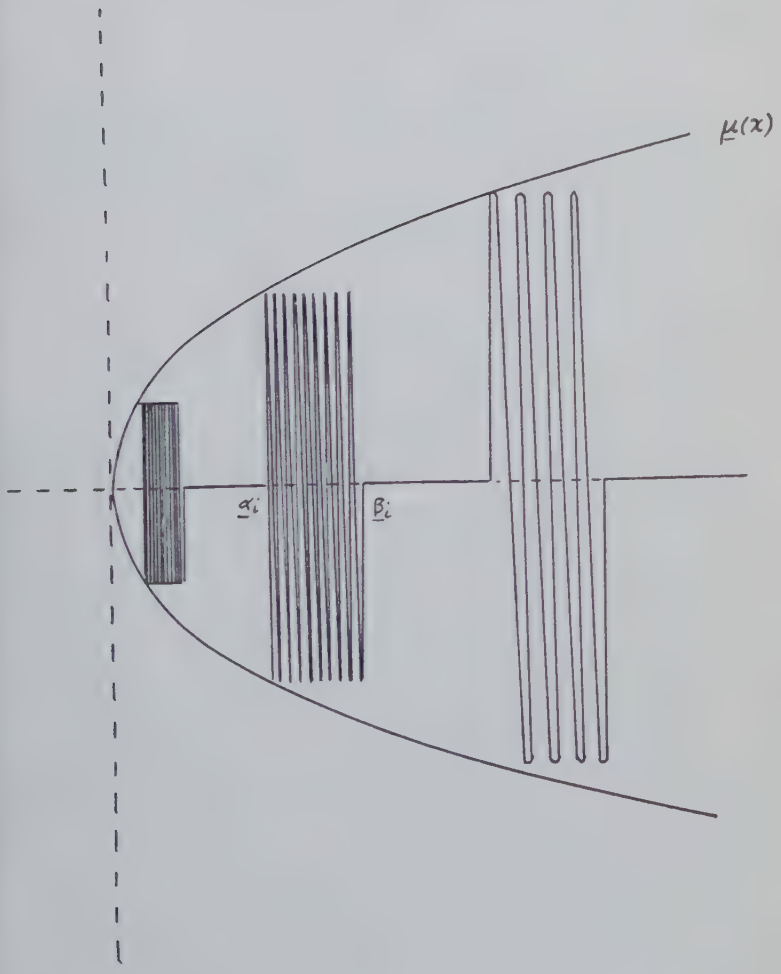
$$\left|\int_{\alpha_i}^{\beta_i} f(x)\sin n_i x\,dx\right| = \left|\int_{\alpha_i}^{\beta_i} k(x)\sin^2 n_i x\,dx\right| \sim k(\alpha_i)\alpha_i$$

since $\beta_i - \alpha_i \sim \alpha_i$.

Now

$$k(\alpha_i)\alpha_i = \frac{\alpha_i}{\left(\log \frac{1}{\alpha_i}\right)^{2/3}} \sim \frac{1}{2^{2i}(\log 2^{2i})^{2/3}}$$

$$\sim \frac{2^i}{n_i(\log n_i)^{2/3}} \sim \frac{(\log n_i)^{1/3}}{n_i}$$



since $\log n_i \sim i + 2^i \sim 2^i$. This is the desired result.

We now show that $f(x)$ is at most of class $\phi(n) = n/(\log n)^{1/3}$. We consider $\int_a^b f(x) \sin nx dx$. The number n may be expressed in the form $n = np + t$ where $0 \leqq t < n_{p+1} - n_p$. Then $n_p \leqq n < n_{p+1}$. We thus have

$$\int_a^b f(x) \sin nx dx = \int_a^b f(x) \sin (n_p + t)x dx.$$

This integral may be expressed as

$$\int_0^b - \int_0^a f(x) \sin (n_p + t)x dx$$

and it is thus sufficient to consider integrals of this type. We write

$$\begin{aligned} &\int_0^b f(x) \sin (n_p + t)x dx \\ (22) \qquad &= \sum_{i=m}^{\infty} \int_{\alpha_i}^{\gamma_i} f(x) \sin (n_p + t)x dx \\ &= \sum_{i=m}^{\infty} \int_{\alpha_i}^{\gamma_i} k(x) \sin n_i x \sin (n_p + t)x dx \end{aligned}$$

where $\alpha_i < \gamma_i \leqq \beta_i$; $m \geqq 1$. The right member of (22) may be written as

$$(23) \qquad \sum_{i=m}^{p-1} + \sum_{i=p}^{\infty} \int_{\alpha_i}^{\gamma_i} k(x) \sin n_i x \sin (n_p + t)x dx.$$

The second sum of (23) may be written as

$$\begin{aligned} &\int_{\alpha_p}^{\gamma_p} k(x) \sin n_p x \sin (n_p + t)x dx \\ (24) \qquad &+ \sum_{i=p+1}^{\infty} \int_{\alpha_i}^{\gamma_i} k(x) \sin n_i x \sin (n_p + t)x dx. \end{aligned}$$

In this sum we have

$$\sum_{i=p+1}^{\infty} \left| \int_{\alpha_i}^{\gamma_i} k(x) \sin n_i x \sin (n_p + t)x dx \right| \sim \sum_{p+1}^{\infty} k(\alpha_i)\alpha_i,$$

and this does not exceed

$$k(\alpha_{p+1}) \sum_{p+1}^{\infty} \alpha_i \sim k(\alpha_{p+1})\alpha_{p+1}.$$

But

$$k(\alpha_{p+1})\alpha_{p+1} \sim \frac{(\log n_{p+1})^{1/3}}{n_{p+1}},$$

which does not exceed a quantity of the order

$$\frac{\log (n_p + t)^{1/3}}{n_p + t} = \frac{(\log n)^{1/3}}{n}.$$

In considering the first term of (24) we distinguish two cases:

Case I. $t < n_p$. We write

$$\left| \int_{\alpha_p}^{\gamma_p} k(x) \sin n_p x \sin (n_p + t)x dx \right| \sim k(\alpha_p)\alpha_p.$$

Hence

$$k(\alpha_p)\alpha_p \sim \frac{(\log n_p)^{1/3}}{n_p} \sim \frac{(\log (n_p + t))^{1/3}}{n_p + t} = \frac{(\log n)^{1/3}}{n}.$$

For, since $t < n_p$, $n_p + t \sim n_p$.

Case II. $t \geqq n_p$. In this case we write

$$\left| \int_{\alpha_p}^{\gamma_p} k(x) \sin n_p x \sin (n_p + t)x dx \right|$$

$$= \left| \int_{\alpha_p}^{\gamma_p} k(x) \cos tx dx - \int_{\alpha_p}^{\gamma_p} k(x) \cos (2n_p + t)x dx \right|. \tag{25}$$

For the second term in the right member of (25) we have

$$\left| \int_{\alpha_p}^{\gamma_p} k(x) \cos (2n_p + t)x dx \right| \sim \frac{k(\alpha_p)}{2n_p + t} \sim \frac{k(\alpha_p + t)}{n_p + t}$$

$$\leqq \frac{(\log (n_p + t))^{1/3}}{n_p + t} \sim \frac{(\log n)^{1/3}}{n}$$

$$\frac{k(\alpha_p)}{n_p} < k(\alpha_p)\alpha_p.$$

The first integral in (25) gives us

$$\left|\int_{\alpha_p}^{\gamma_p} k(x)\cos txdx\right| = O\left(\frac{k(\gamma_p)}{t}\right) \sim \frac{k(\alpha_p)}{t}.$$

But here we have $t \geqq n_p$, so that $t \sim n_p + t$. In fact we may say that $t \geqq (n_p+1)/2$. Thus, from the above relation,

$$\left|\int_{\alpha_p}^{\gamma_p} k(x)\cos txdx\right| \leqq \frac{2k(\alpha_p)}{n_p+t} \sim \frac{k(\alpha_p+t)}{n_p+t} \sim \frac{(\log(n_p+t))^{1/3}}{n_p+t} = \frac{(\log n)^{1/3}}{n}.$$

Thus far we have the desired result for the infinite sum in (23). We now prove it for the finite part. We have

$$\sum_{i=m}^{p-1}\left|\int_{\alpha_i}^{\gamma_i} k(x)\sin n_i x \sin(n_p+t)xdx\right|$$

$$\leqq \sum_{i=1}^{p-1}\left|\int_{\alpha_i}^{\gamma_i} k(x)\cos(n_p+t-n_i)xdx\right| + \sum_{1}^{p-1}\left|\int_{\alpha_i}^{\gamma_i} k(x)\cos(n_p+t+n_i)xdx\right|. \tag{26}$$

The function $k(x)$ plays no part in determining the order here since its greatest value is for the index $i=1$. The first sum in the right member of (26) can then be taken to be

$$\sum_{i=1}^{p-1}\left|\int_{\alpha_i}^{\gamma_i} \cos(n_p+t-n_i)xdx\right| \sim \sum_{i=1}^{p-1}\frac{1}{n_p+t-n_i}.$$

The greatest value of $1/(n_p+t-n_i)$ is attained for $i=p-1$, namely $1/(n_p+t-n_{p-1})$. But

$$\frac{1}{n_p+t-n_{p-1}} \sim \frac{1}{n_p+t}.$$

Hence

$$\sum_{i=1}^{p-1}\frac{1}{n_p+t-n_i} \leqq \frac{p}{n_p+t},$$

and since $p \sim \log \log n_p$, it follows that

$$\sum_{i=m}^{p-1} \left| \int_{\alpha_p}^{\gamma_i} \cos (n_p + t - n_i)x\,dx \right| \leqq \frac{\log \log n_p}{n_p + t} < \frac{(\log (n_p + t))^{1/3}}{n_p + t} = \frac{(\log n)^{1/3}}{n}.$$

In a similar manner we see that the second sum of (26) can be considered in the form

$$\sum_{i=1}^{p-1} \left| \int_{\alpha_i}^{\gamma_i} \cos (n_p + t + n_i)x\,dx \right| \sim \sum_{i=1}^{p-1} \frac{1}{n_p + t + n_i} \leqq \sum_{i=1}^{p-1} \frac{1}{n_p + t} \leqq \frac{p}{n_p + t}.$$

The rest of the argument is as before. The proof is now complete, and we see that $f(x)$ is indeed of class

$$\phi(n) = n/(\log n)^{1/3}.$$

We wish now to determine the modulus of continuity of $f(x)$. We shall consider the function in the interval $[\alpha_i, \beta_i]$. Since we consider max $|f(x_2) - f(x_1)|$, $|x_2 - x_1| < \delta$, it is sufficient to consider only right hand differences. We may write

$$\left| \frac{\Delta f}{\Delta x} \right| < K |f'(x)|$$

where K is a positive constant. We have

$$\begin{aligned} f'(x) &= k(x)n_i \cos n_i x + k'(x) \sin n_i x \\ &= k(x)n_i \cos n_i x \end{aligned}$$

and

$$|f'(x)| \leqq |k(x)n_i| = \mu(\alpha_i)n_i.$$

It is convenient to consider three cases here.

Case I. We consider two points x_1 and $x_2 = x_1 + \delta$ of $[\alpha_i, \beta_i]$ with δ less than a half period of $f(x)$. Then we have $|f(x_2) - f(x_1)| < K n_i \mu(\alpha_i)\delta$. For δ equal to a half period this is of the order $\mu(\alpha_i)$ since then $\delta \sim 1/n_i$. But then $\mu(\alpha_i) \sim \mu(1/n_i) \sim \mu(\delta)$. Since for

a half period $Kn_i\mu(\alpha_i)\delta \sim \mu(\delta)$ it follows that $Kn_i\mu(\alpha_i)\delta = C\mu(\delta)$ and $|\Delta f| < C\mu(\delta)$.

Case II. If x_1 and x_2 are in $[\alpha_i, \beta_i]$ and δ is greater than a half period, then, since $\mu(\delta)$ is an increasing function and Δf can be no greater than the maximum in a half period, the result of Case I holds *a fortiori.*

Case III. If x_1 is in the interval $[\alpha_i, \beta_i]$ and $\delta > \alpha_i = \beta_i - \alpha_i$, then

$$|f(x_2) - f(x_1)| = |f(x_1 + \delta) - f(x_1)| < 2\mu(x_1 + \delta) \sim C\mu(\delta)$$

since $x_1 \sim \alpha_1$ and $\delta > \alpha_1$ so that $x_1 + \delta \sim \delta$.

If we take the point x_1 at the origin, then, since $f(0) = 0$, we have max $|f(x_2) - f(x_1)| = \max |f(x_2)|$ which is max $|f(x_1 + \delta)| = \max |f(\delta)| = \mu(\delta)$.

Thus the modulus of continuity of the function $f(x)$ is of the order

$$\mu(\delta) \sim \frac{1}{\left(\log \frac{1}{\delta}\right)^{2/3}}.$$

From Theorem 6 we have

$$|R_n| \leqq \omega\left(\frac{1}{n}\right)\left[A \log \theta(n) + B\frac{n}{\phi(n)}\right] + \frac{CK}{\theta(n)}$$

with

$$1 \leqq \theta(n) \leqq \phi(n); \qquad 1 \leqq \frac{\theta(n+1)}{\theta(n)} \leqq \frac{\phi(n+1)}{\phi(n)}.$$

For the example

$$\phi(n) = \frac{n}{(\log n)^{1/3}}, \qquad \omega(\delta) = \frac{1}{\left(\log \frac{1}{\delta}\right)^{2/3}}.$$

We may take $\theta(n) = \log n$. Then we have

$$1 \leqq \log n \leqq \frac{n}{(\log n)^{1/3}}$$

and

$$1 \leqq \frac{\log (n+1)}{\log n} \leqq \frac{n+1}{n}\left(\frac{\log n}{\log (n+1)}\right)^{1/3}.$$

Hence

$$|R_n| \leqq \frac{1}{(\log n)^{2/3}}\left[A \log\log n + B(\log n)^{1/3}\right] + \frac{CK}{\log n}$$

$$\leqq \frac{A \log\log n}{(\log n)^{2/3}} + \frac{B}{(\log n)^{1/3}} + \frac{CK}{\log n},$$

and this approaches zero as $n \to \infty$. It follows that the Fourier series of $f(x)$ converges uniformly.

A more general example is obtained if we take

$$\mu(x) = \frac{1}{\left(\log \frac{1}{x}\right)^{s}}, \qquad \phi(n) = \frac{n}{(\log n)^{1-s}}$$

with $s > 1/2$.

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